Here are some optional problems on integral equations. They are taken *verbatim* from Paul Goldbart’s homework sets.

1) **Integral equations:**

   a) Solve the inhomogeneous type II Fredholm integral equation

   \[ u(x) = e^x + \lambda \int_0^1 xy u(y) \, dy. \]

   b) Solve the homogeneous type II Fredholm integral equation

   \[ u(x) = \lambda \int_0^{\pi} \sin(x - y) \, u(y) \, dy. \]

   c) Solve the inhomogeneous type II Fredholm integral equation

   \[ u(x) = x + \lambda \int_0^1 y(x + y) \, u(y) \, dy \]

   to second order in \( \lambda \) using

   i) the Liouville-Neumann-Born series; and

   ii) the Fredholm series.

   d) By differentiating, solve the integral equation: \( u(x) = x + \int_0^x u(y) \, dy. \)

   e) Solve the integral equation: \( u(x) = x^2 + \int_0^1 xy u(y) \, dy. \)

   f) Find the eigenfunction(s) and eigenvalue(s) of the integral equation

   \[ u(x) = \lambda \int_0^1 e^{x-y} u(y) \, dy. \]

   g) Solve the integral equation: \( u(x) = e^x + \lambda \int_0^1 e^{x-y} u(y) \, dy. \)
2) **Neumann Series**: Consider the integral equation

\[ u(x) = g(x) + \lambda \int_0^1 K(x, y) u(y) \, dy, \]

in which only \( u \) is considered unknown.

a) Write down the solution \( u(x) \) to second order in the Liouville-Neumann-Born series.

b) Suppose \( g(x) = x \) and \( K(x, y) = \sin 2\pi xy \). Compute \( u(x) \) to second order in the Liouville-Neumann-Born series. (You may leave your answer to the second-order term in terms of a single integral.)

3) **Translationally invariant kernels**:

a) Consider the integral equation: 
\[ u(x) = g(x) + \lambda \int_{-\infty}^{\infty} K(x, y) u(y) \, dy, \]
with the translationally invariant kernel \( K(x, y) = Q(x-y) \), in which \( g, \lambda \) and \( Q \) are considered known. Show that the Fourier transforms \( \hat{u}, \hat{g} \) and \( \hat{Q} \) satisfy \( \hat{u}(q) = \hat{g}(q) / \{1 - \sqrt{2\pi \lambda} \hat{Q}(q)\} \). Expand this result to second order in \( \lambda \) to recover the second-order Liouville-Neumann-Born series.

b) Use Fourier transforms to find a solution of the integral equation
\[ u(x) = e^{-|x|} + \lambda \int_{-\infty}^{\infty} e^{-|x-y|} u(y) \, dy \]

which remains finite as \( |x| \to \infty \).

c) Use Laplace transforms to find a solution for \( x > 0 \) of the integral equation
\[ u(x) = e^{-x} + \lambda \int_0^x e^{-|x-y|} u(y) \, dy. \]