1) **Critical Mass**: An infinite slab of fissile material has thickness $L$. The neutron density $n(r)$ in the material obeys the equation

$$\frac{\partial n}{\partial t} = D\nabla^2 n + \lambda n + \mu,$$

where $n$ is zero at the surface of the slab at $x = 0, L$. Here $D$ is the neutron diffusion constant, the term $\lambda n$ describes the creation of new neutrons by induced fission, and $\mu$ is the rate of production per unit volume of neutrons by spontaneous fission. Assume that $n$ depends only on $x$ and $t$, and that $\lambda$ and $\mu$ are constants,

a) Expand both $n$ and $\mu$ as series

$$n(x,t) = \sum_m a_m(t) \varphi_m(x), \quad \mu = \sum_m b_m \varphi_m(x)$$

where the $\varphi_m$ are a complete orthonormal set of functions you think suitable for solving the problem.

b) Find an explicit expression for the coefficients $a_m(t)$ in terms of their initial values $a_m(0)$.

c) Determine the critical thickness, $L_{\text{crit}}$, above which the slab will explode.

d) Assuming that $L < L_{\text{crit}}$, find the equilibrium distribution $n_{\text{eq}}(x)$ of neutrons in the slab. (You may either sum your series expansion to get an explicit closed-form answer, or use another (Green function?) method.)

2) **Semi-infinite Rod**: Consider the heat equation

$$\frac{\partial \theta}{\partial t} = D\nabla^2 \theta, \quad 0 < x < \infty$$

with the temperature $\theta(x,t)$ obeying the initial condition $\theta(x,0) = \theta_0$ for $0 < x < \infty$, and the boundary condition $\theta(0,t) = 0$.

a) Show that the boundary condition at $x = 0$ can be satisfied at all times by introducing a suitable mirror image of the initial data in the region $-\infty < x < 0$, and then applying the heat kernel for the entire real line to this extended initial data. Show that the solution of the semi-infinite rod problem can be expressed in terms of the error function

$$\text{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2} d\xi.$$  

b) Solve the same problem by using a Fourier integral expansion in terms of $\sin kx$ on the half-line $0 < x < \infty$ and obtaining the time evolution of the Fourier coefficients. Invert
the transform and show that your answer reduces to that of part a). (Hint: replace the initial condition by \( \theta(x,0) = \theta_0 e^{-\epsilon x} \), so that the Fourier transform converges, and then take the limit \( \epsilon \to 0 \) at the end of your calculation.)

3) 2-D Electron Gas — an old Qual problem:

A two-dimensional gas of electrons is confined at the \( z = 0 \) interface between two semi-infinite dielectric slabs. Each slab has dielectric constant \( \varepsilon \). A perturbation of the electron charge-density propagates as a wave through the electron gas. The surface-charge density on the interface is therefore given by \( \sigma(x,t) = \sigma_0 + \sigma_1(x,t) \), where \( \sigma_0 \) is constant and the small-amplitude perturbation \( \sigma_1 \) takes the form

\[
\sigma_1(x,t) = a \exp\{i(kx - \omega t)\}.
\]

Assume that electrons act as classical particles of mass \( m \) with local velocity,

\[
v(x,t) = v_0 \exp\{i(kx - \omega t)\},
\]

and that the only significant force is due to the electric field produced by the charge density perturbation.

a) Use Laplace’s equation

\[
-\nabla^2 \phi = \varepsilon^{-1} \delta(z) \sigma(x,t)
\]

to find the electrical potential \( \phi(x,z,t) \) due to the charge.

b) From \( \phi(x,z,t) \) find the electric field component \( E_x(x,z = 0,t) \) parallel to and within the electron gas, and hence the acceleration \( \partial v(x,t) / \partial t \) of the electrons.

c) Linearize the charge continuity equation

\[
\frac{\partial \sigma}{\partial t} + \frac{\partial \sigma v}{\partial x} = 0,
\]

and use it to relate \( a \) and \( v_0 \). Hence show that the dispersion equation relating the frequency \( \omega \) to the wavenumber \( k \) is

\[
\omega^2 = \gamma |k|.
\]

Express the coefficient \( \gamma \) in terms of \( m, \varepsilon, \sigma_0 \) and the electron charge \( q = -e \).
4) **Seasonal Heat Waves:** Suppose that the measured temperature of the air above the arctic permafrost is expressed as a Fourier series

\[ \theta(t) = \theta_0 + \sum_{n=1}^{\infty} \theta_n \cos n\omega t, \]

where \( T = \frac{2\pi}{\omega} \) is one year. Solve the heat equation for the soil temperature

\[ \frac{\partial \theta}{\partial t} = \kappa \frac{\partial^2 \theta}{\partial z^2}, \quad 0 < z < \infty \]

with this boundary condition, and find the temperature \( \theta(z, t) \) at a depth \( z \) below the surface as a function of time. Observe that the sub-surface temperature fluctuates with the same period as that of the air, but with a phase lag that depends on the depth. Also observe that the longest period temperature fluctuations penetrate the deepest into the ground. (Hint: for each Fourier component, write \( \theta \) as \( \text{Re}[A_n(z) \exp in\omega t] \) where \( A_n \) is a complex function of \( z \).)