Solutions to Homework Set 1

Differential calculus: The point of the exercise was to make sure that you know how to differentiate integrals with respect to their limits:

$$\frac{d}{da}\int_a^b f(x)\,dx = -f(a), \qquad \frac{d}{db}\int_a^b f(x)\,dx = f(b),$$

and in general

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\,dx = f(b(t))\frac{\partial b}{\partial t} - f(a(t))\frac{\partial a}{\partial t} + \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(x,t)\,dx.$$

Once you have this under control, the first problem is plug and chug:
From

$$y(x) = \frac{\sin\omega(x-L)}{\omega\sin\omega L}\int_0^x f(t)\sin\omega t\,dt + \frac{\sin\omega x}{\omega\sin\omega L}\int_x^L f(t)\sin\omega(t-L)\,dt$$

we get

$$y'(x) = \frac{\cos\omega(x-L)}{\sin\omega L}\int_0^x f(t)\sin\omega t\,dt + \frac{\cos\omega x}{\sin\omega L}\int_x^L f(t)\sin\omega(t-L)\,dt.$$

The two terms arising from the derivative of the integration limits have cancelled against each other. When we differentiate again, the two pieces arising from differentiating the factors outside the integral assemble to give $-\omega^2 y(x)$. The two terms from differentiating the integrals are

$$y''(x) + \omega^2 y(x) = f(x)\frac{\cos\omega(x-L)\sin\omega x - \cos\omega x\sin\omega(x-L)}{\sin\omega L}.$$

The addition formula for $\sin(A+B)$, now shows that this is equal to $f(x)$ as required.

For the second problem we have

$$\begin{aligned}
F'(x) &= K(0)f(x) + \int_0^x \partial_x K(x-y) f(y)\,dy \\
&= K(0)f(x) - \int_0^x f(y)\partial_y K(x-y)\,dy \\
&= K(0)f(x) - \int_0^x \partial_y[f(y)K(x-y)]\,dy + \int_0^x f'(y)K(x-y)\,dy \\
&= K(0)f(x) - K(0)f(x) + f(0)K(x) + \int_0^x f'(y)K(x-y)\,dy \\
&= f(0)K(x) + \int_0^x K(x-y)f'(y)\,dy.
\end{aligned}$$

He was not quite right therefore—unless $f(0)$ happens to be zero.

Improper Integrals: His mistake is at the first step:

$$I(\lambda,\mu)=\int_0^\infty t^{-1}(e^{-\lambda t}-e^{-\mu t})\,dt \stackrel{?}{=} \int_0^\infty t^{-1}e^{-\lambda t}\,dt-\int_0^\infty t^{-1}e^{-\mu t}\,dt.$$

The integrals on the right-hand side are divergent, so this is nonsense. To fix his method we put limits on the divergent integrals:

$$\begin{aligned}
\int_0^\infty \frac{e^{-\lambda t}-e^{-\mu t}}{t}\,dt &= \lim_{\epsilon\to 0}\left\{\int_\epsilon^\infty \frac{e^{-\lambda t}-e^{-\mu t}}{t}\,dt\right\}\\
&= \lim_{\epsilon\to 0}\left\{\int_\epsilon^\infty \frac{e^{-\lambda t}}{t}\,dt-\int_\epsilon^\infty \frac{e^{-\mu t}}{t}\,dt\right\}\\
&= \lim_{\epsilon\to 0}\left\{\int_{\lambda\epsilon}^\infty \frac{e^{-t}}{t}\,dt-\int_{\mu\epsilon}^\infty \frac{e^{-t}}{t}\,dt\right\}\\
&= \lim_{\epsilon\to 0}\left\{\int_{\lambda\epsilon}^{\mu\epsilon} \frac{e^{-t}}{t}\,dt\right\}\\
&= \lim_{\epsilon\to 0}\left\{\int_{\lambda}^{\mu} \frac{e^{-\epsilon s}}{s}\,ds\right\}\\
&= \int_\lambda^\mu \frac{1}{s}\,ds\\
&= \ln\mu-\ln\lambda.
\end{aligned}$$

We could take the limit in the antepenultimate line because $e^{-\epsilon s}\to 1$ uniformly in $[\lambda,\mu]$

Feynman would differentiate under the integral sign:

$$\begin{aligned}
\frac{d}{d\mu}I(\lambda,\mu) &= \int_0^\infty \frac{d}{d\mu}\left(\frac{e^{-\lambda t}-e^{-\mu t}}{t}\right)dt\\
&= \int_0^\infty e^{-\mu t}\,dt\\
&= 1/\mu.
\end{aligned}$$

Integrating up, he would find.

$$I(\lambda,\mu)=\ln\mu+c.$$

He can determine the constant c by noting that $I=0$ when $\lambda=\mu$, whence

$$I(\lambda,\mu)=\ln\mu-\ln\lambda.$$

When f does not tend to zeo at infinity we can use the same scaling trick there to get the general *Frulani integral*

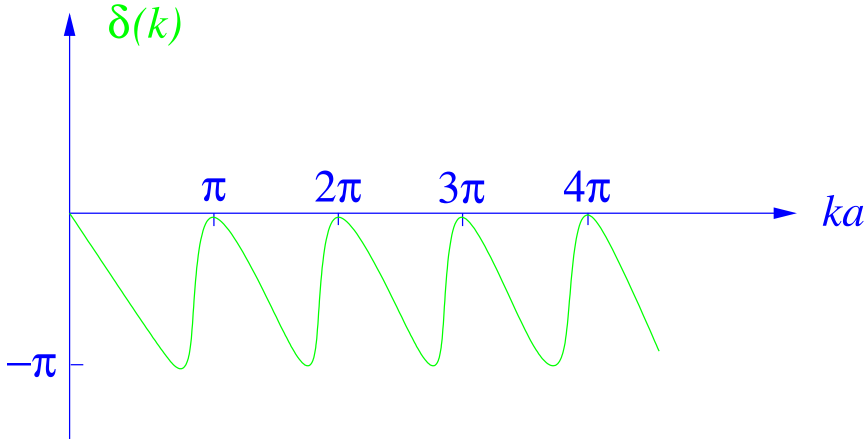
$$F(\lambda,\mu)=\int_0^\infty \frac{f(\lambda t)-f(\mu t)}{t}\,dt=(f(0)-f(\infty))\ln(\mu/\lambda).$$

We apply this to the case $f(x)=\ln(a+be^{-px})$ to get

$$I=\int_0^\infty \ln\left\{\frac{a+be^{-px}}{a+be^{-qx}}\right\}\frac{dx}{x}==\ln\left\{\frac{a}{a+b}\right\}\ln\left\{\frac{p}{q}\right\}$$

This example is found in Gradshteyn and Ryzhik's *Table of Integrals, Series, and Products* as number 4.319.3.

Trigonometry: You need realize that λ is completelely negligeable when $\cot ka$ is close to $\pm\infty$. With no λ, and choosing the branches of the arc-cotangent so as to preserve continuity, we have $\cot^{-1}(\cot ka) = ka$ for all values of ka. The only effect of a large (positive) λ is to make the arc-cotangent hang around $\cot^{-1}(+\infty) = n\pi$ for a while, until $\cot ka$ becomes sufficiently negative to drag the $+\infty$ down to finite values. A plot of $\eta(k) = \cot^{-1}(\lambda + \cot ka)$ is therefore a staircase with flat treads at $\eta = n\pi$ and sharp risers just before $ka = n\pi$, the risers being sharper and closer to $ka = n\pi$ the larger is λ. Adding in the $-ka$ gives:



Sketch plot of $\delta(k) = -ka + \cot^{-1}(\lambda + \cot ka)$.

Mathematica is unable to deal with the infinities, and gives plots containing unphysical discontinuities.

Integration by parts: We'll do the harder parts first.

To recover $\chi(x)$ from $y(x)$ look at

$$\begin{aligned} I(x) &= y(x) + D(x)\int_0^x \frac{D'(\xi)}{D^2(\xi)} y(\xi)d\xi \\ &= \left(\chi(x) - \int_0^x \frac{D'(\xi)}{D(\xi)}\chi(\xi)d\xi\right) + D(x)\int_0^x \frac{D'(\xi)}{D^2(\xi)}\left(\chi(\xi) - \int_0^\xi \frac{D'(\eta)}{D(\eta)}\chi(\eta)d\eta\right)d\xi. \end{aligned}$$

And apply integration by parts to the last term

$$\begin{aligned} J &= -D(x)\int_0^x \frac{D'(\xi)}{D^2(\xi)}\left(\int_0^\xi \frac{D'(\eta)}{D(\eta)}\chi(\eta)d\eta\right)d\xi \\ &= D(x)\int_0^x \left(\frac{1}{D(\xi)}\right)'\left(\int_0^\xi \frac{D'(\eta)}{D(\eta)}\chi(\eta)d\eta\right)d\xi \\ &= -D(x)\int_0^x \left(\frac{1}{D(\xi)}\right)\left(\int_0^\xi \frac{D'(\eta)}{D(\eta)}\chi(\eta)d\eta\right)'d\xi + D(x)\left[\frac{1}{D(\xi)}\int_0^\xi \frac{D'(\eta)}{D(\eta)}\chi(\eta)d\eta\right]_0^x \\ &= -D(x)\int_0^x \frac{D'(\xi)}{D^2(\xi)}\chi(\xi)d\xi + \int_0^x \frac{D'(\xi)}{D(\xi)}\chi(\xi)d\xi. \end{aligned}$$

All but the first term in $I(x)$ cancels, so giving $I(x) = \chi(x)$.

Recovering $y(x)$ from $\chi(x)$ is slightly easier. Look at

$$K(x) = \left(y(x) + D(x) \int_0^x \frac{D'(\xi)}{D^2(\xi)} y(\xi) d\xi \right) - \int_0^x \frac{D'(\xi)}{D(\xi)} \left(y(\xi) + D(\xi) \int_0^\xi \frac{D'(\eta)}{D^2(\eta)} y(\eta) d\eta \right) d\xi$$

and use integration by parts to simplify the last term

$$\begin{aligned} L &= -\int_0^x D'(\xi) \left(\int_0^\xi \frac{D'(\eta)}{D^2(\eta)} y(\eta) d\eta \right) d\xi \\ &= +\int_0^x D(x) \left(\int_0^\xi \frac{D'(\eta)}{D^2(\eta)} y(\eta) d\eta \right)' d\xi - \left[D(\xi) \int_0^\xi \frac{D'(\eta)}{D^2(\eta)} y(\eta) d\eta \right]_0^x \\ &= +\int_0^x \frac{D'(\xi)}{D(\xi)} y(\xi) d\xi - \left[D(\xi) \int_0^\xi \frac{D'(\eta)}{D^2(\eta)} y(\eta) d\eta \right]_0^x \\ &= +\int_0^x \frac{D'(\xi)}{D(\xi)} y(\xi) d\xi - D(x) \int_0^x \frac{D'(\xi)}{D^2(\xi)} y(\xi) d\xi. \end{aligned}$$

Again, all but the first term cancels leaving $K(x) = y(x)$.

If we write the linear map that takes $\chi \to y$ as $y = F(\chi)$, and the map that takes $y \to \chi$ as $\chi = G(y)$ we have shown in part (a) that $F \circ G = \mathrm{Id}$, and in part (b) that $G \circ F = \mathrm{Id}$. (Here $(F \circ G)(x)$ means $F(G(x))$ and Id is the identity map.) If there were two distinct χ_1 and χ_2 such that $y = F(\chi_1)$ and $y = F(\chi_2)$ then $0 = F(\chi_1) - F(\chi_2) = F(\chi_1 - \chi_2)$. Now $G(0) = 0$, so $0 = (G \circ F)(\chi_1 - \chi_2)$ and this is $\mathrm{Id}(\chi_1 - \chi_2) = 0$ or equivalently $\chi_1 - \chi_2 = 0$.

Matrix Algebra:

i) Suppose that T has an eigenvector $\mathbf{x}$ with eigenvalue μ, so $T\mathbf{x} = \mu\mathbf{x}$. Then

$$0 = (T - \lambda I)^N \mathbf{x} = (\mu - \lambda)^N \mathbf{x}.$$

Since $\mathbf{x}$ is non-zero we see that $(\lambda - \mu)^N = 0$, but if the N-th power of a number is zero, the number itself must be zero. Thus λ is the only possible eigenvalue. If the matrix representing T were diagonalizable, all the numbers on the diagonal would have to be λ and the diagonalized matrix would be $\mathbf{T} = \lambda\mathbf{I}$. This matrix, and hence the linear operator T that it represents, would then obey $(\mathbf{T} - \lambda\mathbf{I})^1 = 0$ — but, unless $N = 1$, this is in contradiction to what we were told about T. Thus T *cannot* be diagonalized.

ii) That there exists a vector $\mathbf{e}_1$ such that $(T - \lambda I)^N \mathbf{e}_1 = 0$, but no lesser power of $(T - \lambda I)$ kills $\mathbf{e}_1$, is simply a restatement of what we are given: all vectors are killed by $(T - \lambda I)^N$, but if there were no vector that survived $(T - \lambda I)^{N-1}$, then we would have $(T - \lambda I)^{N-1} = 0$. We are told, however, that $(T - \lambda I)^{N-1}$ is not zero.

iii) A set of vectors $\{\mathbf{e}_i\}$ is *linearly independent* if no non-trivial linear combination of the vectors is the zero vector. We therefore need to show that if

$$\lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 + \ldots + \lambda_N \mathbf{e}_N = 0,$$

then all N of the λ_i are zero.

To do this, act on this expression by $(T - \lambda I)^{N-1}$. We find

$$0 = \lambda_1 (T - \lambda I)^{N-1} \mathbf{e}_1,$$

all other terms being killed. Since we know that $(T-\lambda I)^{N-1}\mathbf{e}_1 \neq 0$, we must have $\lambda_1 = 0$. Now act with $(T-\lambda I)^{N-2}$, to get

$$0 = \lambda_2 (T-\lambda I)^{N-2}\mathbf{e}_2 = \lambda_2 (T-\lambda I)^{N-1}\mathbf{e}_1,$$

all other terms being killed, or being zero because $\lambda_1 = 0$. We now deduce that $\lambda_2 = 0$. Proceeding in this manner we deduce that all the λ_i are zero. Thus the $\mathbf{e}_i$ are indeed linearly independent.

iv) In the $\mathbf{e}_i$ basis the matrix becomes

$$T \to \mathbf{T} = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & 1 & \\ & & & \lambda & \\ & & & & \ddots \end{pmatrix},$$

i.e. a matrix with λ's on the diagonal, and 1's immediately above, all other entries being zero. This is called the *Jordan canonical form* of the non-diagonalizable operator.