Solutions to Homework Set 10

1) Conducting strip:

$$A(k) = \int_{-\infty}^{+\infty} V(x)e^{-ikx}\,dx = V_0 \int_{-a}^{a} e^{-ikx}\,dx = \frac{2V_0 \sin ka}{k}$$

From this we have

$$V(x, 0, y) = 2V_0 \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{\sin(ka)}{k} e^{ikx} e^{-|k||y|}.$$

Taking the y gradients to get E_y, and then getting $\sigma = \epsilon_0(E_y|_{y=+\varepsilon} - E_y|_{y=-\varepsilon})$ gives us

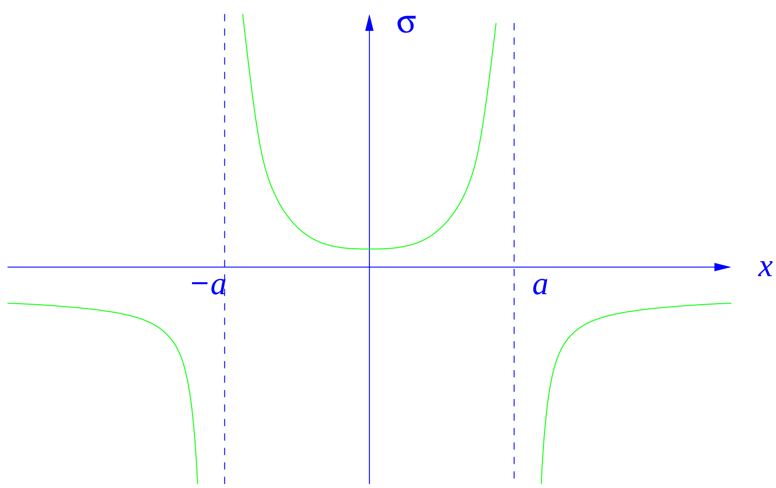
$$\sigma(x) = 4V_0\epsilon_0 \int_{-\infty}^{\infty} \frac{dk}{2\pi} \text{sgn}(k) \sin(ka)\, e^{ikx} e^{-\varepsilon|k|}.$$

The integral is elementary, and gives

$$\sigma(x) = \frac{4V_0\epsilon_0}{4\pi} \left(\frac{1}{x + a - i\varepsilon} - \frac{1}{x - a - i\varepsilon} + \frac{1}{x + a + i\varepsilon} - \frac{1}{x - a + i\varepsilon} \right)$$

We can take safely take the ε regulator to zero. We end up with

$$\sigma(x) = \frac{2\epsilon_0 V_0}{\pi} \left(\frac{1}{a + x} + \frac{1}{a - x} \right) = \frac{4\epsilon_0 V_0}{\pi} \frac{a}{a^2 - x^2}.$$



A sketch of the charge distribution on the strips.

2) Qual Problem:

a) We have, in general

$$b_l = \left(\frac{2l + 1}{2} \right) R^{l+1} \int_{-1}^{1} d(\cos\theta) V(\theta) P_l(\cos\theta).$$

Plugging in the explicit expression for the relevent P_l gives

$$\begin{aligned} b_1 &= \frac{3}{4}R^2(V_1 - V_2) \\ b_2 &= 0 \\ b_3 &= \frac{7}{16}R^4(V_2 - V_1) \end{aligned} \qquad (1)$$

b) If

$$V(r,\theta,\phi) = \frac{1}{4\pi\epsilon_0}\left\{\frac{Q}{r} + \frac{\mathbf{d}\cdot\hat{\mathbf{r}}}{r^2} + \cdots\right\},$$

then Q is the total charge, and $\mathbf{d}$ is the dipole moment. Thus our divided sphere has

$$|d| = 3\pi\epsilon_0 R^2(V_1 - V_2).$$

c) For the sphere immersed in the external field we have

$$V(r,\theta,\phi) = -|E|r\cos\theta + V_{\text{sphere}}(r,\theta,\phi),$$

and so

$$V(r,\theta,\phi) = -|E|r\cos\theta + \frac{1}{4\pi\epsilon_0}\left\{\frac{Q}{r} + \frac{|d|\cos\theta}{r^2} + \cdots\right\}.$$

The surface of the sphere at $r = R$ must be an equipotential, and so the coefficients of the $\cos\theta$ terms must add to zero:

$$-|E|R + \frac{|d|}{4\pi\epsilon_0 R^2} = 0.$$

Thus

$$|d| = 4\pi\epsilon_0 R^3|E|.$$

3) Tides and Gravity:

a) In the non-inertial rotating frame we experience a centrifugal force which can be expressed as $\mathbf{F}_{\text{centrifugal}} = -\nabla\varphi_{\text{centrigugal}}$, where

$$\begin{aligned} \varphi_{\text{centrigugal}} &= -\frac{1}{2}\omega^2(x^2+y^2) = -\frac{1}{2}R^2\sin^2\theta \\ &= \frac{1}{3}\omega^2R^2\Big(P_2(\cos\theta) - 1\Big). \end{aligned}$$

The effective potential experienced by the fluid in the rotating frame is therefore

$$\varphi_{\text{effective}} = \varphi_{\text{gravity}} + \frac{1}{3}\omega^2R^2\Big(P_2(\cos\theta) - 1\Big).$$

b) The tidal force is due to the inhomogeneity of the gravitational potential of the distant star. We work in a frame in which the position of the star and planet are fixed. This is a non-inertial frame: It is rotating once per year. The gravitational potential due to the star is then

$$\begin{aligned} \varphi_{\rm star} &= -\frac{GM_{\rm star}}{|\mathbf{r}_0 - \mathbf{R}|} \\ &= -\frac{GM_{\rm star}}{r_0} \sum_{l=0}^{\infty} \left(\frac{R}{r_0}\right)^l P_l(\cos\theta). \end{aligned}$$

Here $\mathbf{r}_0$ is a vector from the centre of the planet to the centre of the star, and $\mathbf{R}$ is a vector from the centre of the planet to the point of interest on its surface. Assume that $M_{\rm star} \gg m_{\rm planet}$. This condition allows us to ignore reduced-mass effects. Since the orbit is circular, the orbital period and radius are linked by

$$\frac{GM_{\rm star} m_{\rm planet}}{r_0^2} = m_{\rm planet} r_0 \Omega^2.$$

Form this equation we obtain

$$\frac{GM_{\rm star}}{r_0^3} = \Omega^2.$$

The $l = 0$ term is a constant and can be dropped. The $l = 1$ term cancels agains the centrifugal potential due to the orbital motion — provided that we regard the orbital centrifugal force as being constant throughout the body of the planet. The *tidal force* is therefore derived from

$$\varphi_{\rm tide} = -R^2 \Omega^2 P_2(\cos\theta).$$

Observe that r_0 no longer appears, so we do not have to know the distance to the star.

The centrifugal force is not really constant on the scale of the planet, but its variation is entirely due to the once-per year rotation of the planet about its own axis (as opposed to the reference frame's rotation about the axis through the centre of the star). To convince yourself of this without doing a calculation, consider a planet which is still orbiting in a circle but is not rotating in the inertial frame of the "fixed stars". It is easy to see that every point in the body this non-rotating planet is in a circular orbit with the same period and radius, but with a different centre. Consequently, every point in the non-rotating planet has the same centripetal acceleration vector. From the point of view of the non-inertial frame in which it is at rest, every point in the body of the planet therefore experiences the same centrifugal force.

Now allow the planet to rotate once per year so as to keep the same face towards the star. This once-per-year rotation should be lumped in with the much larger diurnal axial rotation so as to get a total angular velocity whose direction defines the polar axis of the planet. The annual rotation therefore contributes to the oblate equatorial bulge about the polar axis, and not to the prolate tidal deformation, whose symmetry axis points to the star.
d) The potential is continuous at $R = R_0$, so

$$\frac{A_2}{R_0^3} = B_2 R_0^2.$$

From

$$4\pi G\rho_0\eta P_2(\cos\theta) = \left.\frac{\partial\varphi}{\partial R}\right|_+ - \left.\frac{\partial\varphi}{\partial R}\right|_- ,$$

we have

$$4\pi G\rho_0\eta = -\frac{3A_2}{R_0^4} - 2B_2R_0^2.$$

Thus

$$A_2 = -\frac{4\pi}{5}G\rho_0R_0^4\eta, \quad B_2 = -\frac{4\pi}{5}G\rho_0R_0^{-1}$$

e) We will focus on the tidal deformation. The equatorial bulge problem is a trivial modification of this. The basic fact underlying both deformations is that planetary surface, being the surface of a fluid, must be an equipotential.
The total potential is the sum of three terms:

i) The external potential $\varphi_{\text{tide}} = -\Omega^2R_0^2P_2(\cos\theta)$. This may be evaluated at R_0 because it is already first order in the perturbation.

ii) The potential φ_{shell} of the shell. This may also be evaluated at R_0, and we can use either the inner or outer potential to get $\varphi_{\text{shell}} = -4\pi G\rho_0R_0\eta P_2(\cos\theta)/5$.

iii) The potential φ_{sphere} of the undeformed planet. Here we do have to take into acount the variation in height of the bulge. Thus

$$\varphi_{\text{sphere}} = -\frac{4}{3}\frac{\pi\rho_0R_0^3G}{(R_0 + P_2(\cos\theta)\eta)} = \text{const.} + \frac{4}{3}\pi\rho_0R_0G\eta P_2(\cos\theta).$$

The sum of the three terms must be independent of angle, and so the coefficient of $P_2(\cos\theta)$ must be zero. Thus

$$0 = -\Omega^2R_0^2 + 4\pi G\rho_0R_0\eta\left(-\frac{4}{5} + \frac{4}{3}\right),$$

leading to

$$\eta_{\text{tide}} = \frac{15}{2}\frac{\Omega^2R_0}{4\pi G\rho_0}.$$