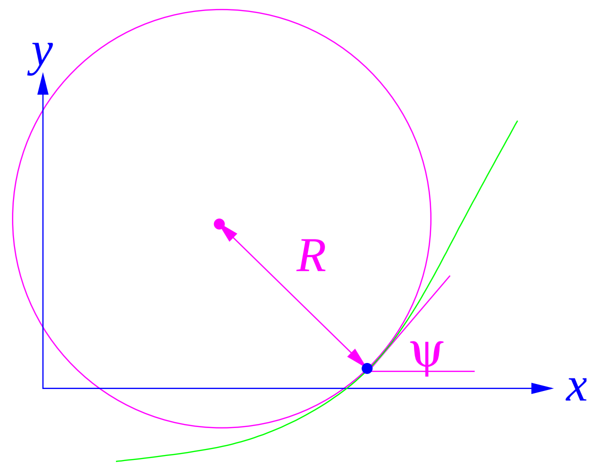
Solutions to Homework Set 3

1) Bent bars: First recall some elementary calculus results: the curvature κ, and the radius R of the osculating circle, at a point (x, y) on a curve $y(x)$ are given by

$$\kappa = \frac{1}{R} = \frac{d\psi}{ds} = \frac{1}{\sqrt{1+y'^2}} \frac{d\tan^{-1}y'}{dx} = \frac{y''}{(1+y'^2)^{3/2}}.$$

Here the arc length s and the angle ψ between the curve and the x axis constitute the *intrinisic coordinates* of the curve.



Osculating circle and intrinsic coordinates.

For all parts of the present problem y' is assumed small enough that we can get away with the approximation $R^{-1} = y''$.

a) Euler's problem: We observe that if a rod of fixed length L is bent into a curve $y(z)$, we have

$$ds^2 = dz^2 + dy^2, \quad \Rightarrow \quad dz = \sqrt{1-\dot{y}^2}\, ds = \left(1 - \frac{1}{2}\dot{y}^2 + \cdots\right) ds.$$

Here the overdot indicates differentiation with respect to the arc-length s. For curves with small y', we may approximate $y' = \dot{y}$ and $ds \approx dz$ in the last expression. The length of the projection of the rod on the z axis is therefore

$$L - \int_0^L \frac{1}{2} y'^2 dz.$$

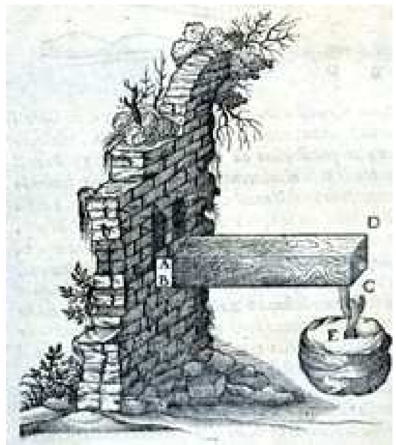
Combining the resulting expresion for the potential energy of the load with the bending energy gives the functional $U[y]$.

Inserting the mode expansion for $y(z)$ and doing the integral gives

$$U[y] = \frac{L}{2} \sum_{n=1}^{\infty} a_n^2 \left\{ \frac{YI}{2}\left(\frac{n^4\pi^4}{L^4}\right) - \frac{Mg}{2}\left(\frac{n^2\pi^2}{L^2}\right) \right\}.$$

The coefficient of a_n^2 is negative when $Mg > n^2\pi^2 YI/L^2$. We thus see that the coefficient of a_1^2 is the first to go negative, making $n = 1$ mode unstable, and this occurs at when $Mg = \pi^2 YI/L^2$.

b) Leonardo da Vinci's problem: The strength of a cantilever was discussed by Galileo in his *Dialogue Concerning Two New Sciences* published 1654. His reasoning was not quite complete however. Surprisingly, a more correct analysis of the problem had been given by Leonardo da Vinci more than a century earlier.



The drawing from Galileo's Dialogue.

The energy is

$$E[y] = \int_0^L \frac{YI}{2}(y'')^2\,dx + Mgy(L).$$

This is a minimum when

$$0 = \delta E = \int_0^L YIy^{(4)}\delta y(x)\,dx + YI[y''\delta y' - y'''\delta y]_0^L + Mg\,\delta y(L)$$

for any variation $\delta y(x)$. At the wall end, $x = 0$, we are told that $y(0) = y'(0) = 0$. The displacement therefore satisfies $y^{(4)} = 0$, with these boundary conditions. Thus, we know that the minimum energy configuration will be of the form

$$y(x) = ax^2 + bx^3,$$

but do not yet know the constants a and b. Now consider the integrated-out terms at the weight end:

$$YIy''(L)\delta y'(L) - (YIy'''(L) - Mg)\delta y(L).$$

Even though we have made $\int_0^L YIy^{(4)}\delta y(x)\,dx$ vanish, we can still vary $y(L)$ and $y'(L)$ *independently* by varying the as-yet-unknown constants a and b:

$$\begin{aligned} \delta y(L) &= L^2\delta a + L^3\delta b, \\ \delta y'(L) &= 2L\,\delta a + 3L^2\delta b. \end{aligned}$$

The variations are independent because given any desired $\delta y(L)$ and $\delta y'(L)$, we can solve for the necessary δa, δb as

$$\begin{aligned} \delta a &= 3L^{-2}\delta y(L) - 2L^{-3}\delta y'(L), \\ \delta b &= -L^{-1}\delta y(L) + L^{-2}\delta y'(L). \end{aligned}$$

Therefore, in the minimum-energy configuration, the coefficients of $\delta y(L)$ and $\delta y'(L)$ must vanish separately, giving the boundary conditions

$$\begin{aligned} y''(L) &= 0, \\ YIy'''(L) &= Mg. \end{aligned}$$

The unique solution is

$$y(x) = \frac{Mg}{YI}\left(\frac{1}{6}x^3 - \frac{1}{2}Lx^2\right).$$

Thus $y(L) = -\frac{1}{3}MgL^3/YI$. Note that

$$YIy''(x) = Mg(x - L).$$

The right-hand-side of this last equation is the *bending moment* of the weight on the section of the rod at x. The maximum bending moment occurs right where the rod enters the wall, and this is where the cantilever would break were the weight too large.

Lagrange multipliers. Recall that given a symmetric matrix $\mathbf{T}$ the problem of finding the stationary points of $\mathbf{x} \cdot \mathbf{Tx}$ subject to the condition $\mathbf{x} \cdot \mathbf{x} = 1$ reduces to the problem of finding the normalized eigenvectors of the matrix $\mathbf{T}$. The given quadratic form is

$$13x^2 + 8xy + 7y^2 = (\,x, \quad y\,)\begin{pmatrix} 13 & 4 \\ 4 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

We therefore have to diagonalize the matrix

$$\mathbf{T} = \begin{pmatrix} 13 & 4 \\ 4 & 7 \end{pmatrix}.$$

It is possible to diagonalize a 2×2 matrix almost by inspection. We begin by observing that product of the eigenvalues is

$$\det \mathbf{T} = 13 \times 7 - 4 \times 4 = 75.$$

The sum of the eigenvalues is

$$\operatorname{tr} \mathbf{T} = 13 + 7 = 20.$$

The eigenvalues are therefore $\lambda = 5$ and $\lambda = 15$. The easiest way to find the eigenvectors is to use Dirac's trick (see the appendix in the notes) to decompose the identity matrix into operators $\mathbf{P}_\lambda$ that project onto the eigenvectors corresponding to eigenvalue λ:

$$\begin{aligned} \mathbf{I} &= \frac{(\mathbf{T} - 5\mathbf{I})}{15 - 5} + \frac{(\mathbf{T} - 15\mathbf{I})}{(5 - 15)} \\ &= \quad \mathbf{P}_{15} \quad + \quad \mathbf{P}_5 \end{aligned} \tag{1}$$

The projection operators are therefore

$$\mathbf{P}_5 = \frac{1}{10}(15\mathbf{I} - \mathbf{T}), \quad \mathbf{P}_{15} = \frac{1}{10}(\mathbf{T} - 5\mathbf{I}).$$

We can project from almost any vector, so let's use $\mathbf{v} = (0,1)^t$. The eigenvector corresponding to $\lambda = 15$ is proportional to

$$(\mathbf{T} - 5\mathbf{I})\mathbf{v} = \begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}.$$

After normalizing we find that there are stationary points at $x = \pm 2/\sqrt{5}$, $y = \pm 1/\sqrt{5}$, at which $f(x,y) = 15$

Similarly, the eigenvector corresponding to $\lambda = 5$ is proportional to

$$(15\mathbf{I} - \mathbf{T})\mathbf{v} = \begin{pmatrix} 2 & -4 \\ -4 & 8 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 8 \end{pmatrix},$$

so the other stationary points are at $x = \mp 1/\sqrt{5}$, $y = \pm 2/\sqrt{5}$, at which $f(x,y) = 5$

Catenary again: We seek to make stationary

$$F[x,y] = \int ds \left\{ \rho g y + \frac{1}{2}\lambda(s)(\dot{x}^2 + \dot{y}^2 - 1) \right\}.$$

Part a): Since the ends of the chain are fixed, we can discard any integrated out terms, and so have

$$\delta F = \int ds \left\{ \delta x(s) \left(-\frac{d}{ds}\left(\lambda(s)\dot{x}\right) \right) + \delta y(s) \left(\rho g - \frac{d}{ds}\left(\lambda(s)\dot{y}\right) \right) \right\}.$$

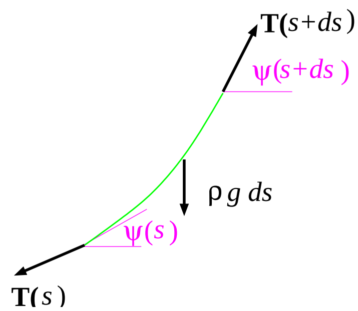
The potential energy is therefore stationary when

$$\begin{aligned} 0 &= \frac{\delta F}{\delta x(s)} \equiv -\frac{d}{ds}\left(\lambda(s)\dot{x}\right) \\ 0 &= \frac{\delta F}{\delta y(s)} \equiv \rho g - \frac{d}{ds}\left(\lambda(s)\dot{y}\right). \end{aligned}$$

Taking into acount the condition $\dot{x}^2 + \dot{y}^2 = 1$, which tells us that s is the arc-length, we can set $\dot{x} = \cos\psi$ and $\dot{y} = \sin\psi$, where ψ is the intrinsic coordinate. Thus

$$\begin{aligned} 0 &= \frac{d}{ds}\left(\lambda(s)\cos\psi\right) \\ \rho g &= \frac{d}{ds}\left(\lambda(s)\sin\psi\right). \end{aligned}$$

The figure shows that these are the conditions for the equilibrium of each infinitesimal segment of the chain, with $\lambda(s) \to T(s)$ being the tension at point s.



The free body diagram for the forces acting on a segment of chain of length ds.

The horizontal component of the force

$$[T(s)\cos\psi]_s^{s+ds} \approx \frac{d}{ds}\left(\lambda(s)\cos\psi\right) ds$$

must vanish, while the vertical component balance is

$$[T(s)\sin\psi]_s^{s+ds} \approx \frac{d}{ds}\left(\lambda(s)\sin\psi\right) ds = \rho g\, ds.$$

Part b): The geometry tells us that $\psi = s/a$. We know that $T(s)\cos\psi$ is a constant, C say, and so

$$\begin{aligned} \rho(s)g &= \frac{d}{ds}\left(C\tan\left(\frac{s}{a}\right)\right) \\ &= \frac{C}{a}\sec^2\left(\frac{s}{a}\right). \end{aligned} \tag{2}$$

The constant C is fixed by making

$$Mg = \int_{-\pi a/4}^{\pi a/4} \frac{C}{a}\sec^2\left(\frac{s}{a}\right) ds = C[\tan\psi]_{-\pi/4}^{\pi/4} = 2C.$$

Thus

$$\rho(s) = \frac{M}{2a}\sec^2\left(\frac{s}{a}\right).$$