

Solutions to Homework Set 6

1) Missing state:

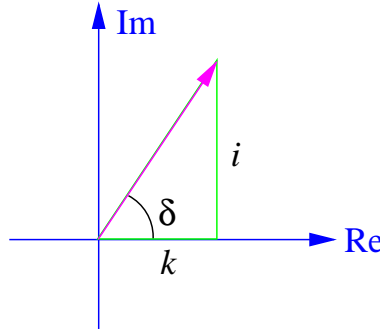
- The continuum eigenfunctions are are

$$\psi_k = (-ik + \tanh x)e^{ikx},$$

so

$$\psi_k(x) = \begin{cases} (k - i)e^{ikx}/i & x \ll 0, \\ (k + i)e^{-ikx}/i & x \gg 0. \end{cases}$$

It now helps to draw a phasor diagram



from which we see that $\delta(k) = \tan^{-1}(1/k)$ and $A = -i\sqrt{1+k^2}$.

- The periodic boundary conditions require

$$Ae^{-i\delta}e^{-ikL/2} = Ae^{i\delta}e^{ikL/2},$$

so

$$2\pi n = 2\delta(k) + kL.$$

The density of states is therefore given by

$$\rho(k) = \frac{dn}{dk} = \frac{1}{2\pi} \left(L + 2\frac{\partial\delta}{\partial k} \right).$$

- The free density of states is $\rho_0 = L/2\pi$, so

$$\Delta N = \int_{-\infty}^{\infty} \{ \rho(k) - \rho_0(k) \} dk = \int_{-\infty}^{\infty} \left\{ 2\frac{\partial\delta}{\partial k} \right\} \frac{dk}{2\pi} = \frac{1}{\pi} [\delta(k)]_{-\infty}^{\infty}.$$

Now when $k \rightarrow -\infty$ we have $\delta \rightarrow +\pi$, and when $k \rightarrow +\infty$ we have $\delta \rightarrow 0$. Thus

$$\Delta N = -1,$$

and there is one fewer state in the continuum than there was when there was no potential. The lowest energy continuum state was peeled away from the others, and has become localized as the bound state $\psi_0 = \frac{1}{\sqrt{2}}\text{sech } x$.

2) Continuum completeness:

- For positive κ

$$\psi_0(x) = \sqrt{2\kappa}e^{-\kappa x}$$

is a normalized eigenstate of the given operator with eigenvalue $E = -\kappa^2$. It obeys the boundary condition

$$\psi/\psi'|_{x=0} = \tan \theta = -\kappa^{-1}.$$

The solution is not normalizable if κ is negative, and so there is no bound state if $\tan \theta$ is positive.

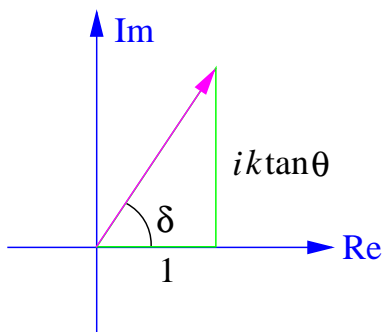
- The state

$$\psi_k(x) = \sin(kx + \delta(k))$$

is an eigenfunction with eigenvalue $E = k^2$. The boundary condition demands that

$$\psi/\psi'|_{x=0} = \tan \theta = \frac{1}{k} \tan \delta(k).$$

Thus $\tan \delta(k) = k \tan \theta$ and, again most easily by drawing a phasor diagram



we find that

$$e^{i\delta(k)} = \frac{1 + ik \tan \theta}{\sqrt{1 + k^2 \tan^2 \theta}}.$$

Take care with arc-tangents: $\tan^{-1}(x/y)$ appears to be the same as $\tan^{-1}(-x/-y)$, but when these formulæ represent the polar angle for the point (x, y) they must differ by π . We fixed the π ambiguity in δ (reflected by the two possible signs for the square root), by requiring δ to be zero when $\tan \theta = 0$.

- The expected completeness condition is

$$\psi_0(x)\psi_0(y) + \frac{2}{\pi} \int_0^\infty \psi_k(x)\psi_k(y) dk = \delta(x - y).$$

- Substituting in $\psi_k(x)$, using the explicit expression for $\delta(k)$ and standard trig identities, we find

$$\frac{2}{\pi} \int_0^\infty \psi_k(x)\psi_k(y) dk = \int_{-\infty}^\infty e^{ik(x-y)} \frac{dk}{2\pi} - \int_{-\infty}^\infty \frac{dk}{2\pi} e^{ik(x+y)} \left\{ \frac{2(1 + ik \tan \theta)}{1 + k^2 \tan^2 \theta} - 1 \right\}.$$

We have also used $e^{i\delta(-k)} = e^{-i\delta(k)}$ to piece together the second integral on the right hand side.

The integral

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x+y)}$$

can only contribute if $x = y = 0$, so can be ignored.

Using the given standard integral, we have

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x+y)} \frac{2}{1+k^2 \tan^2 \theta} = \frac{1}{|\tan \theta|} e^{-|x+y|/|\tan \theta|},$$

and

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x+y)} \frac{2ik \tan \theta}{1+k^2 \tan^2 \theta} = -\operatorname{sgn}(x+y) \frac{\tan \theta}{|\tan \theta|^2} e^{-|x+y|/|\tan \theta|},$$

the latter integral being obtained from the former by differentiating with respect to x so as to bring down a factor of ik from the exponent. Since x, y are both positive, $\operatorname{sgn}(x+y) = +1$.

Putting the parts together, we find that

$$\frac{2}{\pi} \int_0^{\infty} \psi_k(x) \psi_k(y) dk = \int_{-\infty}^{\infty} e^{ik(x-y)} \frac{dk}{2\pi} - \frac{1}{|\tan \theta|} \left(1 - \frac{\tan \theta}{|\tan \theta|} \right) e^{-|x+y|/|\tan \theta|}.$$

The first term on the right gives $\delta(x-y)$. The second term is equal to

$$-2\kappa e^{-\kappa x} e^{-\kappa y} = -\psi_0(x) \psi_0(y)$$

when $\tan \theta$ is negative, and is zero when $\tan \theta$ is positive. The conjectured completeness condition is therefore confirmed. Enjoy the delicious intricacy of the cancellation that eliminates the bound state once it ceases to be normalizable!

3) Fredholm Alternative:

- Looking back at the solutions to homework set 4, we see that

$$\int_0^1 u^* v^{(4)} dx = [u^* v^{(3)} - (u')^* v'' + (u'')^* v' - (u^{(3)})^* v]_0^1 + \int_0^1 (u^{(4)})^* v dx.$$

If the boundary conditions defining $\mathcal{D}(\hat{L})$ are $v'' = v''' = 0$ then the integrated out part will vanish for all such v provided we take $u'' = u''' = 0$ as the domain of the adjoint operator. This is the same as the domain of the original operator, and so \hat{L} is self-adjoint.

- The functions $1, x, x^2$ and x^3 all have vanishing fourth derivative, but only the first two satisfy the boundary conditions. The null space of \hat{L} is therefore spanned by $y_1(x) = 1$ and $y_2(x) = x$.
- In order for a solution to $\hat{L}y = f - mg$ to exist, the source term must be orthogonal to the all functions in the null space of \hat{L} . We therefore require that

$$\int_0^L (f(x) - mg) dx = 0, \quad \int_0^L x(f(x) - mg) dx = 0.$$

The first condition says that the net vertical force on the rod must be zero, and the second that the total moment of the applied forces must vanish. Both conditions are familiar from statics.

- If the total applied force and moment vanishes, then the rod will be in equilibrium at any vertical height (addition of any multiple of y_1) and tilt (addition of any multiple of y_2). Of course we assumed that the tilt was small in deriving the equations, so we should not tilt too much.