Solutions to Homework Set 7

1) Flexible rod again: This is a somewhat tedious, but ultimately rewarding exercise.

a) Look back at your solution for homework set 4, where you showed that

$$\int_0^1 u^* v^{(4)}\, dx = [u^* v^{(3)} - (u')^* v'' + (u'')^* v' - (u^{(3)})^* v]_0^1 + \int_0^1 (u^{(4)})^* v\, dx.$$

From this we see that taking $v = v'' = 0$ at the endpoints as the boundary conditions for L requires $u = u'' = 0$ at the endpoints for the as the boundary conditions for $L^\dagger$. The operator is therefore self-adjoint.

b) There are no zero modes for L. The solution is therefore unique, and F has no conditions to satisfy

c) From $\partial_x^4 G(x,y) = \delta(x-y)$ we see that we need G, G' and G'' to be continuous at $x = y$, and G''' must obey the jump condition

$$G'''(y_+, y) - G'''(y_-, y) = 1.$$

The Green function, considered as a function of x, must also satisfy the boundary conditions at $x = 0, 1$.

d) We satisfy the boundary conditions at the ends by writing

$$G(x,y) = \begin{cases} ax + bx^3, & 0 < x < y < 1, \\ c(x-1) + d(x-1)^3, & 0 < y < x < 1. \end{cases}$$

The jump condition, the continuity at $x = y$ of G'', G', and G (in that order) then read:

$$\begin{aligned} 6d - 6b &= 1, \\ 6d(y-1) &= 6by, \\ c + 3d(y-1)^2 &= a + 3by^2, \\ c(y-1) + d(y-1)^3 &= ay + by^3. \end{aligned}$$

The first two equations quickly give $d = y/6$ and $b = (y-1)/6$. Rather more effort produces $c = y(y-1)(y+1)/6$ and $a = y(y-1)(y-2)/6$. Thus

$$G(x,y) = \begin{cases} \frac{1}{6}[y(y-1)(y-2)x + (y-1)x^3], & 0 < x < y < 1, \\ \frac{1}{6}[y(y-1)(y+1)(x-1) + y(x-1)^3], & 0 < y < x < 1. \end{cases}$$

e) To see that that the Green function does indeed obey $G(x,y) = G(y,x)$ it helps to rearrange the second alternative so as to expose the powers of y:

$$G(x,y) = \begin{cases} \frac{1}{6}[y(y-1)(y-2)x + (y-1)x^3], & 0 < x < y < 1, \\ \frac{1}{6}[x(x-1)(x-2)y + (x-1)y^3], & 0 < y < x < 1. \end{cases}$$

On noting that interchanging x and y in $G(x,y)$ not only interchanges the symbols in the above expressions but also the two alternatives (make sure you understand why this is so), we now have manifest $x \leftrightarrow y$ symmetry. This is what we expect for a real, self-adjoint Green function. Observing this symmetry emerge after some very non-symmetric (unless you are cleverer than me in organising your calculation) algebra is the main pleasure of the problem.

f) Plugging in, we find.

$$y(x) = \frac{1}{6}\int_0^x [y(y-1)(y+1)(x-1)+y(x-1)^3]F(y)\,dy + \frac{1}{6}\int_x^1 [y(y-1)(y-2)x+(y-1)x^3]F(y)\,dy.$$

I will not show how differentiating this four times recovers $F(x)$, as this is a purely mechanical exercise.

2) Hot ring: This is a problem with periodic boundary conditions. We can regard the ring as being the entire real line, but with the functions in the domain of $\hat{L} \equiv -\partial_x^2$ being subject to the periodicity condition $u(x+1) = u(x)$. By differentiating this condition n times and then setting $x = 0$, you may be tempted to conclude that u and all its derivatives must take the same values at $x = 0$ and $x = 1$. In fact we may only conclude that $u(0) = u(1)$ and $u'(0) = u'(1)$, which are the boundary conditions specified in the problem. This is because the source function $f(x)$ may not be continuous at the join $(x = 0) \sim (x = 1)$. If f *is* discontinuous, $u''(x) = f(x)$ will take different values at $x = 0_+$ and $x = 1_-$, and "setting $x = 0$" will not result in a unique value for u''.

a) The zero mode is $u_0(x) = const$. The source function f must therefore obey $\int_0^1 f\,dx = 0$. This means that there is no net flux of heat into the ring — a physically obvious condition for there to be steady-state solution to the heat equation.

b) At $x = 0$ we have $|x - y| = y$, so

$$g(0, y) = \frac{1}{2}y^2 - \frac{1}{2}y.$$

At $x = 1$ we have $|x - y| = 1 - y$, so

$$g(1, y) = \frac{1}{2}(1 - y)^2 - \frac{1}{2}(1 - y),$$

which is equal to $g(0, y)$. Similarly

$$g'(0, y) = -y + \frac{1}{2},$$

and

$$g'(1, y) = (1 - y) - \frac{1}{2},$$

so $g'(1, y) = g'(0, y)$. Finally we have

$$-\partial_x^2 \left(\frac{1}{2}(x - y)^2 - \frac{1}{2}|x - y|\right) = -1 + \delta(x - y),$$

So $g(x, y)$ is indeed a suitable modified Green function, and, provided that f satisfies the condition of part a),

$$u(x) = u_0 + \int_0^1 g(x, y)f(y)\,dy,$$

is a solution to $\hat{L}u = f$, for any constant u_0.

3) Lattice Green functions: We use the usual convention that indices increase downward in a column vector. Thus, in a $k \times k$ matrix $\mathbf{T}$ the entry t_{11} is in the top left and t_{kk} in the bottom right.

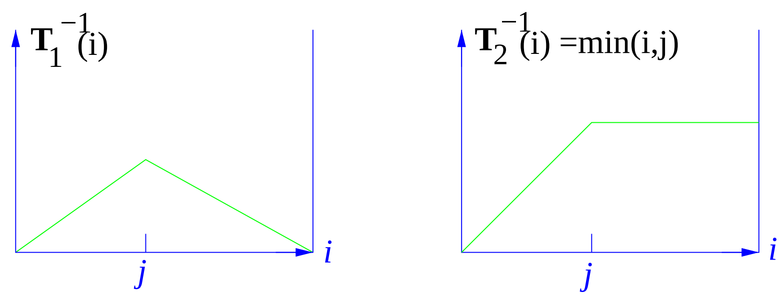
a) The matrix $\mathbf{T}_1$ corresponds to to the Dirichlet boundary conditions at each end: $u_0 = u_{k+1} = 0$. It is the same matrix that appeared in an earlier homework. The matrix $\mathbf{T}_2$ effectively sets $u_{k+1} = u_k$ so that

$$-u_{k+1} + 2u_k - u_{k-1} \to u_k - u_{k-1}.$$

The resulting condition $u_{k+1} - u_k = 0$ corresponds to Neumann ($y' = 0$) boundary conditions at $x = 1$. The boundary condition at $x = 0$ remains Dirichlet.

b) The slope discontinuity of $\min(i,j)$ at $i = j$ makes both of the putative inverses output δ_{ij} when plugged into $-u_{i+1} + 2u_i - u_{i-1}$. It is therefore only necessary to check the boundary conditions that rewrite the first ($i = 0$) and last ($i = k$) equation. We see that $\min(i,j) - ij/(k+1)$ is zero when $i = 0, k+1$ and $1 \le j \le k$. The inverse $\mathbf{T}_1^{-1}$ is correct therefore. Similarly $\min(i,j)$, as a function of i, is constant when $i \ge j$. Because $1 \le j \le k$, it takes the same value at $i = k$ and $i = k+1$, and so $\mathbf{T}_2^{-1}$ also works.

c) These discrete Green functions exactly parallel the continuum expressions:



4) Eigenfunction expansion: From homework set 0 we have

$$y(x) = \frac{\sin\omega(x-L)}{\omega\sin\omega L}\int_0^x f(t)\sin\omega t\,dt + \frac{\sin\omega x}{\omega\sin\omega L}\int_x^L f(t)\sin\omega(t-L)\,dt,$$

and so

$$G_{\omega^2}(x,y) = \frac{1}{\omega\sin\omega L} \times \begin{cases} \sin\omega(x-L)\sin\omega y, & y < x, \\ \sin\omega x\sin\omega(y-L), & y > x. \end{cases}$$

a) As ωL approaches $n\pi$ we see that the $\sin\omega L$ in the denominator is vanishing as

$$\sin\omega L \approx (-1)^n(\omega L - n\pi).$$

The points $\omega^2 = \omega_n^2 \equiv n^2\pi^2/L^2$ are therefore singularities of G_{ω^2}. They are, of course, the eigenvalues of the differential operator.

b) At the same time, the two expressions that apply depending on which of x and y is the greater become proportional. This is because

$$\sin\omega(x-L) \to \sin(\omega_n x - n\pi) = (-1)^n \sin\omega_n x.$$

Thus, when ω^2 is very close to ω_n^2, we have

$$\begin{aligned} G_{\omega^2}(x,y) &\approx \frac{1}{\omega_n(\omega L - n\pi)} \sin\omega_n x \sin\omega_n y \\ &= \frac{1}{2\,\omega_n(\omega - n\pi/L)} \sqrt{\frac{2}{L}} \sin\omega_n x \sqrt{\frac{2}{L}} \sin\omega_n y \\ &\approx \frac{1}{(\omega^2 - n^2\pi^2/L^2)} \sqrt{\frac{2}{L}} \sin\omega_n x \sqrt{\frac{2}{L}} \sin\omega_n y \\ &= \frac{1}{(\omega^2 - \omega_n^2)} \psi_n(x)\psi_n(y), \end{aligned}$$

where

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\omega_n x$$

is the normalized eigenfunction. The residue is therefore the product of the normalized eigenfunctions, as promised.
(My solution has an overall minus sign compared to what you were expected to show, but this is because the G_{ω^2} I borrowed from homework set 0 is the Green function to *minus* the differential operator in this homework set!)