Solutions to Homework Set 8

1) Critical mass: We expand

$$n(x,t) = \sum_{m=1}^{\infty} a_m(t) \sin\left(\frac{m\pi x}{L}\right),$$

and also

$$\mu = \sum_{m,\text{odd}} \frac{4\mu}{m\pi} \sin\left(\frac{m\pi x}{L}\right), \quad 0 < x < L.$$

Substituting in the given equation, and using the linear independence of the sine functions, then gives

$$\dot{a}_m(t) = \left(\lambda - \frac{Dm^2\pi^2}{L^2}\right) a_m(t) + \frac{4\mu}{m\pi},$$

where the last term is present only when m is odd. Let us define

$$\alpha_m = \left(\lambda - \frac{Dm^2\pi^2}{L^2}\right).$$

The solution to the evolution equation is either

$$a_m(t) = \left(a_m(0) + \frac{4\mu}{m\pi\alpha_m}\right) e^{\alpha_m t} - \frac{4\mu}{m\pi\alpha_m},$$

or

$$a_m(t) = a_m(0)e^{\alpha_m t},$$

depending on whether m is odd or even. The $m = 1$ mode is the first to go unstable, and this happens as soon as $\alpha_1 > 0$, *i.e.* when $L > L_{\text{crit}}$ where

$$L_{\text{crit}} = \pi\sqrt{\frac{D}{\lambda}}.$$

To find the equilibrium distribution we solve

$$D\frac{d^2n}{dx^2} + \lambda n + \mu = 0,$$

with the boundary conditions $n(0) = n(L) = 0$. This is an inhomogenenous ODE with constant coefficients. It is therefore most easily solved by combining a complementary function

$$n_{\text{CF}}(x) = A\cos\sqrt{\lambda/D}(x - L/2),$$

which has been chosen to be symmetric about the midpoint of the slab, with the particular integral

$$n_{\text{PI}}(x) = -\frac{\mu}{\lambda}.$$

To satisfy the boundary conditions we take

$$n_{\text{equilibrium}}(x) = \frac{\mu}{\lambda}\left(\frac{\cos\sqrt{\lambda/D}(x-L/2)}{\cos\sqrt{\lambda/D}(L/2)} - 1\right).$$

Note that this blows up (quite literally!) when $\cos\sqrt{\lambda/D}(L/2) = 0$ or when

$$\frac{\pi}{2} = \sqrt{\frac{\lambda}{D}}\frac{L}{2},$$

which is another way of determining that $L_{\text{crit}} = \pi\sqrt{D/\lambda}$.

2) Semi-infinite rod: We solve this problem by an image trick. If we compute the temperature evolution under

$$\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial x^2}$$

of a rod that is infinite in *both* directions, but with initial data $\theta(x,0) = -1$ for $x < 0$ and $\theta(x,0) = 1$ for $x > 0$, then symmetry guarantees that $\theta(0,t) = 0$ for all later times. The solution to this modified problem can be written down by using the heat kernel:

$$\begin{aligned}
\theta(x,t) &= \sqrt{\frac{1}{4\pi t}}\left(\int_0^\infty e^{-(x-\xi)^2/4t}\,d\xi - \int_{-\infty}^0 e^{-(x-\xi)^2/4t}\,d\xi\right) \\
&= \sqrt{\frac{1}{4\pi t}}\int_0^\infty \left(e^{(x-\xi)^2/4t} - e^{-(x+\xi)^2/4t}\right)\,d\xi \\
&= \sqrt{\frac{1}{4\pi t}}\left(\int_{-x}^\infty e^{-\zeta^2/4t}\,d\zeta - \int_x^\infty e^{-\zeta^2/4t}\,d\zeta\right) \\
&= \sqrt{\frac{1}{4\pi t}}\left(\int_{-x}^0 e^{-\zeta^2/4t}\,d\zeta - \int_x^0 e^{-\zeta^2/4t}\,d\zeta\right) \\
&= \frac{2}{\sqrt{4\pi t}}\int_0^x e^{-\zeta^2/4t}\,d\zeta \\
&= \frac{2}{\sqrt{\pi}}\int_0^{x/\sqrt{4t}} e^{-\zeta^2}\,d\zeta \\
&= \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right). \qquad (1)
\end{aligned}$$

The solution to the homework problem is therefore

$$\theta(x,t) = \theta_0 \operatorname{erf}\left(\frac{x}{2\sqrt{Dt}}\right).$$

We used this solution when we discussed Duhamel's method

If we follow the instructions for part b) we find that the Fourier integral solution is

$$\theta(x,t) = \lim_{\epsilon\to 0}\left\{\frac{2}{\pi}\int_0^\infty a(k,t)\sin kx\,dk\right\},$$

where

$$a(k,t) = a(k,0)e^{-k^2 t},$$

and

$$a(k,0) = \int_0^\infty e^{-\epsilon x} \sin kx \, dx = \frac{1}{k+i\epsilon}.$$

The integral over k has no actual singularity at $k=0$, so the $i\epsilon$ can be put safely to zero, whence

$$\theta(x,t) = I(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin kx}{k}\, e^{-k^2 t}\, dk.$$

We now need to evaluate this integral. We observe that I is manifestly equal to zero at $x=0$, and that

$$\frac{dI}{dx} = \frac{2}{\pi}\int_0^\infty \cos kx \, e^{-k^2 t}\, dk = \frac{1}{\pi}\int_{-\infty}^\infty e^{ikx} e^{-k^2 t}\, dk = \sqrt{\frac{1}{\pi t}} \exp\left\{-\frac{1}{4t}x^2\right\}.$$

Integrating up, $I(x) = \int_0^x (dI/d\xi)\, d\xi$, we therefore find that

$$\theta(x,t) = \sqrt{\frac{1}{\pi t}} \int_0^x \exp\left\{-\frac{1}{4t}\xi^2\right\} d\xi = \sqrt{\frac{2}{\pi}} \int_0^{x/\sqrt{4t}} \exp\left\{-\zeta^2\right\} d\zeta = \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right).$$

3) 2DEG: An appropriate solution to Laplace's equation is

$$\phi(x,z,t) = b\, e^{i(kx-\omega t)} e^{-|kz|},$$

for some constant b. From Gauss' law we have

$$\begin{aligned} \sigma_1 \equiv a\, e^{i(kx-\omega t)} &= -\varepsilon(\partial_z\phi|_{z=+\epsilon} - \partial_z\phi|_{z=-\epsilon}) \\ &= 2|k| b\varepsilon\, e^{i(kx-\omega t)}. \\ &\Rightarrow b = a/2|k|\varepsilon. \end{aligned}$$

The electric field is $E_x(x,0,t) = -\partial_x \phi(x,0,t) = -ikb\, e^{i(kx-\omega t)}$, and so Newton's equation of motion $m\partial_t v = qE_x$ gives

$$-i\omega m v_0\, e^{i(kx-\omega t)} = -iqkb\, e^{i(kx-\omega t)} \quad \Rightarrow \quad m\omega v_0 = qbk.$$

From the linearized continuity equation $\partial_t \sigma_1 + \sigma_0 \partial_x v = 0$, we have

$$-i\omega a + ik\sigma_0 v_0 = 0 \quad \Rightarrow \quad v_0 = \omega a / k\sigma_0.$$

Putting it all together we have

$$\frac{m\omega^2 a}{k\sigma_0} = \frac{qka}{2|k|\varepsilon} \quad \Rightarrow \quad \omega^2 = \frac{q|k|\sigma_0}{2m\varepsilon}.$$

Note that although $q = -e$ is a negative number, so is $\sigma_0 = -e\times$(electron density). The product $q\sigma_0$ is therefore *positive*. This ensures that frequency ω is a real number.

4) Seasonal Heat Waves: An E&M analogue of this problem was on the Qual Exam a couple of years ago.

As suggested, we write θ as the real part of $\theta(z)e^{in\omega t}$. Plug into the heat equation and find that

$$in\omega\theta(z) - \kappa\theta''(z) = 0.$$

The two linearly independent solutions to this ODE are

$$\theta(z) = \exp\left\{\pm\sqrt{\frac{in\omega}{\kappa}}z\right\} = \exp\left\{\pm\sqrt{\frac{n\omega}{2\kappa}}(1+i)z\right\}.$$

We must chose the minus sign, so that the effects of the heat fluctuations die away as the depth z becomes large. Thus

$$\theta(z) = \text{Re}\left[\exp\left\{-\sqrt{\frac{n\omega}{2\kappa}}(1+i)z + in\omega t\right\}\right]$$

Matching these solutions to the boundary conditions at $z = 0$, we find

$$\theta(z) = \theta_0 + \sum_{n=1}^{\infty}\theta_n \exp\left\{-\sqrt{\frac{n\omega}{2\kappa}}z\right\}\cos\left(\sqrt{\frac{n\omega}{2\kappa}}z - n\omega t\right).$$

These are damped waves propagating downwards into the ground.