Solutions to Homework Set 9

1) Pantograph Drag: The cable supports waves with dispersion equation $\omega^2 = \Omega^2 + c^2k^2$. Here $c^2 = T/\rho$. The phase velocity ω/k is always $> c$, while the group velocity $\partial\omega/\partial k = c^2k/\omega$ is always $< c$.

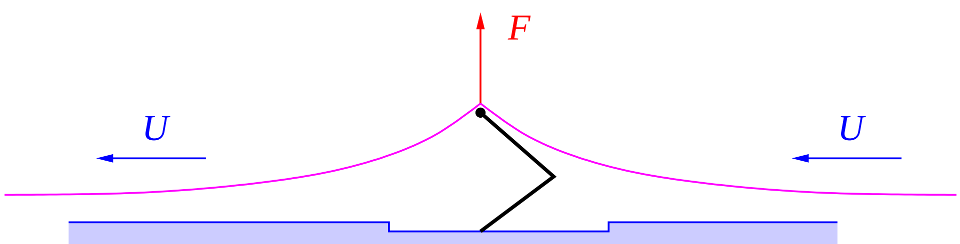
a) We seek a solution of the form $y(x-Ut)$. Plugging this into the equation, we find that the function $y(x)$ must obey

$$-y'' + \frac{\gamma^2\Omega^2}{c^2}y = \frac{\gamma^2 F}{T}\delta(x-Ut).$$

Here $\gamma^2 = (1-U^2/c^2)^{-1}$ is positive, and so the solutions to the homogeneous equation are hyperbolic functions. The pantograph has no effect far from the train so we need to choose exponentially decaying solutions. Using the continuity and jump conditions at the delta function, we find.

$$y(x,t) = \frac{\gamma^2 F}{T}\left(\frac{c}{2\Omega\gamma}\right)\exp\left\{-\frac{\Omega\gamma}{c}|x-Ut|\right\}.$$

The overhead cable is pushed up symmetrically about the point of contact of the pantograph.



Cable profile for $U < c$

As U approaches c from below γ will diverge to infinity, and our small-amplitude assumption will fail.

b) We again look for a solution of the form $y(x-Ut)$, but $U > c$ and so y obeys

$$-y'' - \frac{\tilde\gamma^2\Omega^2}{c^2}y = \frac{\tilde\gamma^2 F}{T}\delta(x-Ut),$$

where now it is $\tilde\gamma^2 = (U^2/c^2-1)^{-1}$ that is positive, and so the solutions to the homogeneous equations are oscillating functions. The locomotive is travelling faster than the group velocity of the waves. The cable is therefore unaware of the approaching train, and is undisturbed until the pantograph pick-up has passed. The solution will thus have $y=0$ for $x > Ut$, and oscillate thereafter. The relevent solution is therefore

$$y(x,t) = \begin{cases} \frac{\tilde\gamma^2 F}{T}\left(\frac{c}{\Omega\tilde\gamma}\right)\sin\left(\frac{\Omega\tilde\gamma}{c}(Ut-x)\right), & x < Ut, \\ 0, & x > Ut. \end{cases}$$

This is the case illustrated by the drawing above the problem. As a check, observe that for this solution $\omega \equiv \Omega\tilde\gamma U/c$ and $k \equiv \Omega\tilde\gamma/c$, obey the dispersion relation $\omega^2 = \Omega^2 + c^2k^2$, and so the wake is indeed a solution of the Klein-Gordon equation.

c) The energy density in the wake is

$$\begin{aligned}\mathcal{E} &= \frac{1}{2}\rho\dot{y}^2 + \frac{1}{2}T(y')^2 + \frac{1}{2}\rho\Omega^2 y^2 \\ &= \rho\left(\frac{\tilde{\gamma}^2 F}{T}\right)^2 U^2 \sin^2\left(\frac{\Omega\tilde{\gamma}}{c}|x-Ut|\right).\end{aligned}$$

The spatial average of $\sin^2 x$ is $1/2$, so the average energy per unit length of wake is

$$\langle\mathcal{E}\rangle = \frac{1}{2}\rho\tilde{\gamma}^4\left(\frac{F}{T}\right)^2 U^2.$$

The rate of working by the engine is UF_{drag} and this must be equal to the rate at which energy is being deposited in the wake. This is $\langle\mathcal{E}\rangle(U-U_g)$. Here

$$U_g = \frac{\partial\omega}{\partial k} = \frac{c^2 k}{\omega}$$

is the group velocity. Since the phase velocity of the wave constituting the wake ω/k is equal to U, we have $U_g = c^2/U$. Therefore

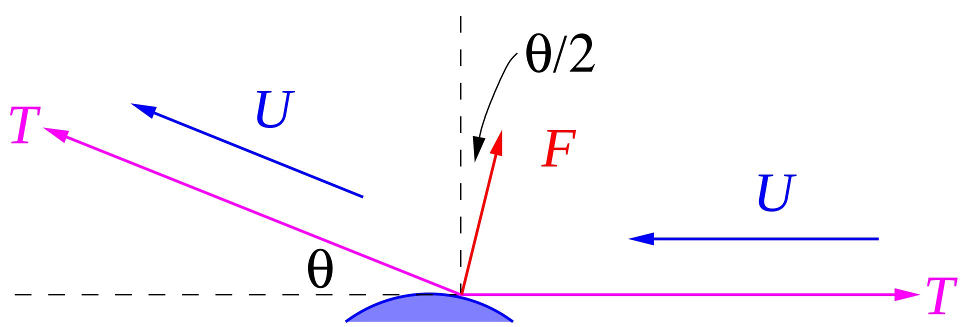
$$UF_{drag} = \langle\mathcal{E}\rangle\frac{c^2}{U}\left(\frac{U^2}{c^2}-1\right) = \frac{1}{2}\frac{F\tilde{\gamma}^2}{T}U,$$

and

$$F_{drag} = \frac{1}{2}\frac{F^2\tilde{\gamma}^2}{T}.$$

This expression for the drag is only valid if $U > c$. There is no wave drag when $U < c$ because there is no wake.

d) It is always a good idea to verify sophisticated computations by appeal to elementary physical principles. We therefore appeal to "$F = ma$" as a reality check: consider, in the frame of the locomotive, the impact of the cable on the pantograph pickup. The cable is striking the pickup and being deflected through an angle θ.



Elastic impact of cable on pickup.

In the absence of friction the tension is unchanged by the impact[1]. Both the resultant of the tension and the acceleration vector of the cable lie symmetrically between the

[1] It is not quite true that T and U are unchanged after the collision with the pantograph, but their changes, due to the cable stretching, are of second order in the amplitude of the wake, and so have a second order effect on the direction of F. This effects a *third order* change in the computed wave drag. The stretching therefore does not affect our computation, which is of amplitude squared accuracy

parts of the cable before and after the impact. The force exerted by the pickup, which contributes to the acceleration, must therefore also lie in this direction. We see that the impact can be thought of as an elastic collision of the cable with the pickup surface, the angle of incidence equalling the angle of reflection. The rate of change of the component of momentum of the cable perpendicular to the pickup is the product of the mass impacting per second, ρU, times the change in perpendicular velocity, which is $2U \sin\theta/2$. Equating this rate of momentum change to the resultant of the applied forces, we have

$$\rho U^2 2 \sin\theta/2 = 2T \sin\theta/2 + F,$$

or, using $c^2 = T/\rho$ and $\tilde{\gamma}^2 = (U^2/c^2 - 1)^{-1}$,

$$F = T\tilde{\gamma}^{-2} 2 \sin\theta/2.$$

Therefore

$$2\sin\theta/2 \approx \theta = \tilde{\gamma}^2 \frac{F}{T}.$$

Observe that this deflection angle is consistent with the initial slope of our wake solution

$$\begin{aligned} y(x,t) &= \frac{\tilde{\gamma}^2 F}{T}\left(\frac{c}{\Omega\tilde{\gamma}}\right)\sin\left(\frac{\Omega\tilde{\gamma}}{c}(Ut - x)\right) \\ &\approx \tilde{\gamma}^2 \frac{F}{T}(-x), \quad x < 0, \quad x \text{ small.} \end{aligned}$$

The component of the normal force $\mathbf{F}$ in the direction of motion of the train is

$$F_{drag} = F \sin\theta/2 = \frac{1}{2}\tilde{\gamma}^2 \frac{F^2}{T}.$$

This coincides with the drag force we computed in part c).

2) Non-linear waves: This particular equation of state holds for, *inter alia*, the one-dimensional non-interacting electron gas. The lack of interaction between the left- and right-going waves can be understood as a descendent property of the lack of interactions between the electrons.

a) Inserting the given equation of state into the hydrodynamic equations, we find

$$\begin{aligned} \partial_t \rho + \rho \partial_x v + v \partial_x \rho &= 0, \\ \partial_t v + v \partial_x v + \lambda \rho \partial_x \rho &= 0. \end{aligned}$$

If we add these, we find

$$\Big(\partial_t + (\lambda\rho + v)\partial_x\Big)(\lambda\rho + v) = 0.$$

If we subtract, we find

$$\Big(\partial_t + (-\lambda\rho + v)\partial_x\Big)(-\lambda\rho + v) = 0.$$

We are now asked to investigate how this pair of equations reduce to the usual sound-wave equation in some limit. If the fluid is at rest in the absence of the sound wave, we can write $v = \epsilon v_1$ and $\rho = \rho_0 + \epsilon\rho_1$, where ρ_0 is the position independent background density and $\epsilon\rho_1$ and ϵv_1 are small amplitude perturbations. The $O(\epsilon)$ mass conservation law and Euler's equation are then

$$\begin{aligned} \partial_t\rho_1 + \rho_0\partial_x v_1 &= 0, \\ \partial_t v_1 + \lambda\rho_0\partial_x\rho_1 &= 0. \end{aligned}$$

Differentiate the first of these with respect to t. Mutiply the second by ρ_0 and differentiate with respect to x. Eliminating the common term $\rho\partial^2_{xt}v_1$, we find find that

$$\partial^2_{tt}\rho_1 - \lambda^2\rho_0^2\,\partial^2_{xx}\rho_1 = 0.$$

This is the acoustic wave equation with

$$c^2_{\text{sound}} = \left.\frac{dP}{d\rho}\right|_{\rho=\rho_0} = \lambda^2\rho_0^2.$$

The first equality holds for any equation of state.
If we write $v_1 = \partial_x\phi$, then Euler's equation in the form

$$\partial_t v + \frac{1}{\rho}\left(\frac{dP}{d\rho}\right)\partial_x\rho = 0$$

reduces at $O(\epsilon)$ to

$$\partial_t v_1 + \left(\frac{c^2}{\rho_0}\right)\partial_x\rho_1 = 0,$$

and this tells us that

$$\rho_1 = -\left(\frac{\rho_0}{c^2}\right)\partial_t\phi.$$

(We used this in our discussion of blast waves in the lectures notes.) Using this result togther with $c = \lambda\rho_0$, we see that in the small amplitude limit we have

$$\Big(\partial_t + (\lambda\rho + v)\partial_x\Big)(\lambda\rho + v) \to \Big(\partial_t + (\lambda\rho_0)\partial_x\Big)\left(-c^{-1}\partial_t\phi + \partial_x\phi\right),$$

and so the first Riemann equation reduces to one of the factorized forms of the wave equation

$$(\partial_t + c\partial_x)(\partial_t\phi - c\partial_x\phi) = \left(\partial^2_{tt} - c^2\partial^2_{xx}\right)\phi = 0.$$

The second Riemann equation reduces to the wave equation with the linear factors in the other order.

b) We see that the full non-linear Riemann equations describe disturbances propagating at the speeed $v \pm c$, where $c = \lambda\rho$ is the local speed of sound in the small amplitude limit. The waves are therefore travelling at the speed of sound, but also being advected by the local velocity field.

c) This is explained in the notes.

3) Burgers' shocks:

a) The wave is moving to the right with speed u. If u decreases to the right, the larger u part will overtake the smaller u part. See drawings in lecture notes.

b) Start from Burgers' equation

$$\partial_t u + u\partial_x u = \nu \partial^2_{xx} u$$

and set $u = \partial_x h$ so that

$$\partial_x \left(\partial_t h + \frac{1}{2}(\partial_x h)^2 - \nu \partial^2_{xx} h \right) = 0.$$

Thus

$$\partial_t h + \frac{1}{2}(\partial_x h)^2 - \nu \partial^2_{xx} h = g(t).$$

We can absorb the $g(t)$ by making the replacement $h \to h + \int^t g(\tau)\, d\tau$, which does not affect u. In this way we set $g(t) \equiv 0$. We next observe that the Riccati-like terms can be combined as

$$\frac{1}{2}(\partial_x h)^2 - \partial^2_{xx} h = 2\nu^2 e^{h/2\nu} \partial^2_{xx} e^{-h/2\nu}.$$

Thus

$$\partial_t h + 2\nu^2 e^{h/2\nu} \partial^2_{xx} e^{-h/2\nu} = 0,$$

or

$$e^{-h/2\nu} \partial_t h + 2\nu^2 \partial^2_{xx} e^{-h/2\nu} = 0.$$

Now

$$e^{-h/2\nu} \partial_t h = -2\nu \partial_t e^{-h/2\nu},$$

so, finally,

$$\partial_t (e^{-h/2\nu}) - \nu \partial^2_{xx} (e^{-h/2\nu}) = 0.$$

Thus, if $u = -2\nu \partial_x \ln \psi$, and ψ obeys the heat equation

$$\partial_t \psi = \nu \partial^2_{xx} \psi,$$

then u obeys Burgers' equation.

c) The given expression

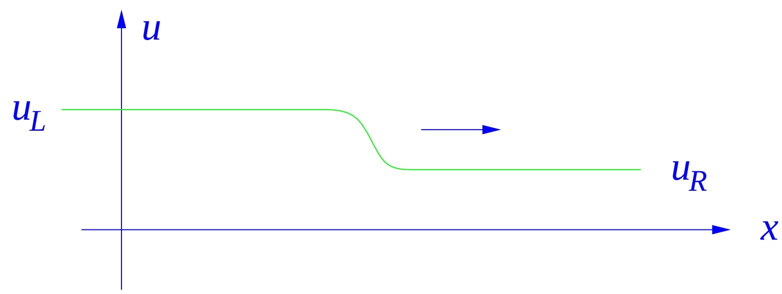
$$\psi = A e^{\nu a^2 t - ax} + B e^{\nu b^2 t - bx}$$

is obviously a solution of the heat equation. In order for $\ln \psi$ to be never singular, both A and B must be positive. Assuming that this is so, then, by shifting the origin of the x co-ordinate, we can set $A = B$. We therefore assume that we have done this. We can now write

$$\psi = A e^{\frac{\nu}{2}(a^2+b^2)t - \frac{1}{2}(a+b)x} 2\cosh\left(\frac{\nu}{2}(a^2 - b^2)t - \frac{1}{2}(a-b)x \right).$$

Forming the logarithmic derivative, we find that

$$u(x,t) = \nu(a+b) + \nu(a-b)\tanh\left(\frac{\nu}{2}(a^2-b^2)t - \frac{1}{2}(a-b)x\right).$$



This is a "shock" wave interpolating between $u_L = 2\nu a$ and $u_R = 2\nu b$, and, if $a^2 > b^2$, is propagating to the right at speed

$$U = \frac{\nu(a^2-b^2)}{(a-b)} = \nu(a+b) = \frac{1}{2}(u_L + u_R).$$

The spatial width of the shock is $2/(a-b) = 4\nu/(u_L - u_R)$.