Essential Calculus Review

We will make heavy use of the following interactions between derivatives and integrals:

a) Differentiating an integral with respect to the upper limit:

\[ \frac{d}{db} \int_a^b f(y) \, dy = f(b). \]

b) Integrating a derivative

\[ \int_a^b \frac{d}{dx} f(x) \, dx = f(b) - f(a). \]

c) Differentiating under the integral sign

\[ \frac{d}{dx} \int_a^b f(y, x) \, dy = \int_a^b \frac{\partial}{\partial x} f(y, x) \, dy. \]

d) Leibniz’ integral rule is a generalization of the last two equations:

\[ \frac{d}{dt} \int_{a(t)}^{b(t)} f(x, t) \, dx = \frac{db}{dt} f(b) - \frac{da}{dt} f(a) + \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(x, t) \, dx. \]

We can combine (a) with Leibniz’ rule for derivatives

\[ \frac{d}{dx} \left( f(x)g(x) \right) = \frac{df(x)}{dx} g(x) + f(x) \frac{dg(x)}{dx} \]

to get the integration-by-parts identity

\[ f(b)g(b) - f(a)g(a) = \int_a^b \frac{df(x)}{dx} g(x) \, dx + \int_a^b f(x) \frac{dg(x)}{dx} \, dy. \]

This is often written as

\[ [fg]^b_a - \int_a^b f'g \, dx = \int_a^b fg' \, dx. \]

Integration by parts is one of the most useful tools in calculus.