Physics 508 | **Mathematical Methods in Physics I** | Prof. M. Stone
Handout MT | `courses.physics.illinois.edu/phys508/fa2019/` | 2117 ESB
Nov 6th 2019 | **Midterm Exam** | University of Illinois

Do all **three** *problems. Check each step before proceeding as there will be no propagation of errors. Marks will be subtracted for any equation that is obvious nonsense.*

1) Green Function: Consider the homogeneous boundary value problem

$$-y'' + m^2 y = f(x), \quad y(a) = y(b) = 0, \quad b > a. \qquad (\star)$$

Here m^2 is a positive constant.

a) Make use of hyperbolic functions such as $\sinh mx$ or $\cosh mx$ to write down solutions $y_L(x)$ and $y_R(x)$ that can be used to construct a Green function for this problem. [4 points]
b) Compute the Wronskian W of your y_L and y_R. Verify that your answer is compatible with Liouville's formula for dW/dx as applied to $(\star)$. [4 points]
c) Construct the explicit Green function appropriate to this problem. [4 points]
d) Use your Green function to write down the solution of the boundary value problem as the sum of two explicit integrals over complementary components of the interval $[a, b]$. [4 points]
e) Confirm that your solution $y(x)$ obeys both boundary conditions, and, by differentiating, that it does indeed solve the original problem. [4 points]

Useful:

$$\begin{aligned} \sinh(-A) &= -\sinh A, \\ \sinh(A - B) &= \sinh A \cosh B - \sinh B \cosh A. \end{aligned}$$

2) Distributions: By making use of the idea of *test functions*

a) Show that
$$f(x)\delta'(x) = f(0)\delta'(x) - f'(0)\delta(x). \quad \text{[6 points]}$$
b) Explain what is meant by the *weak derivative* of a function or distribution. [4 points]
c) As an illustration of a weak derivative, derive the equation
$$\frac{d}{dx} \ln|x| = P\left(\frac{1}{x}\right). \quad \text{[10 points]}$$

The marks in this question are for the *logic* of what is being done, so you must write clear sentences explaining what you are doing, and why, at each step.

— *Exam continues on next page* —

3) Orthogonality and Completeness: The Macdonald functions $\mathrm{K}_\lambda(x)$ with purely imaginary index $\lambda = i\mu$ are real-valued when $0 < x < \infty$, and obey $\mathrm{K}_{i\nu}(x) = \mathrm{K}_{-i\nu}(x)$. They also possess the orthogonality property

$$\frac{1}{\pi^2}\int_0^\infty \frac{dx}{x}\mathrm{K}_{i\mu}(x)\mathrm{K}_{i\nu}(x) = \frac{\delta(\mu-\nu)}{2\nu\sinh\nu\pi}.$$

(You do not need to know anything about Macdonald functions other than what you are told in the statement of this problem!)

a) Assuming that these functions form a complete set for expanding out functions on $x > 0$, write down the *completeness relation* that expresses this fact. [10 points]

b) Given a function $f(x)$ defined for $x > 0$, we form its *Kontorovich-Lebedev* transform $\tilde{f}(\nu)$ by

$$\tilde{f}(\nu) = \int_0^\infty \mathrm{K}_{i\nu}(x) f(x)\, dx.$$

Write down the expression for the inverse transform that allows us to recover $f(x)$ from $\tilde{f}(\nu)$. [10 points]

— *End of Exam* —