Q1 Contour Integration: Use the calculus of residues to evaluate the following integrals:

\[ I_1 = \int_0^{2\pi} \frac{d\theta}{(a + b \cos \theta)^2}, \quad 0 < b < a. \]
\[ I_2 = \int_0^{2\pi} \frac{\cos^2 \theta}{1 - 2a \cos 2\theta + a^2} d\theta, \quad 0 < a < 1. \]
\[ I_3 = \int_0^{\infty} \frac{x^\alpha}{(1 + x^2)^2} dx, \quad -1 < \alpha < 2. \]

(These are not meant to be easy! You will have to dig for the residues.)

Q2 Lattice Matsubara sums: Show that, for suitable functions \( f(z) \), the sum

\[ S = \frac{1}{N} \sum_{\omega} f(\omega) \]

of the values of \( f(z) \) at the \( N \)-th roots of \( -1 \) can be written as an integral

\[ S = \frac{1}{2\pi i} \int_C \frac{dz}{z} \frac{z^N}{z^N + 1} f(z). \]

Here \( C \) consists of a pair of oppositely oriented concentric circles. The annulus formed by the circles should include all the roots of unity, but exclude all singularities of \( f \). Use this result to show that, for \( N \) even,

\[ \frac{1}{N} \sum_{n=0}^{N-1} \frac{\sinh E}{\sinh^2 E + \sin^2 \frac{(2n+1)\pi}{N}} = \frac{1}{\cosh E} \tanh \frac{NE}{2}. \quad (\star) \]

Take the \( N \to \infty \) limit while scaling \( E \to 0 \) in some suitable manner, and hence show that

\[ \sum_{n=-\infty}^{\infty} \frac{a}{a^2 + [(2n + 1)\pi]^2} = \frac{1}{2} \tanh \frac{a}{2}. \quad (\star\star) \]

Take care not to get this last result wrong by a factor of two: it is not true that the limit of the finite sum \((\star)\) is the infinite sum \((\star\star)\).

Q3 Plemelj and Neumann: The Legendre function of the second kind \( Q_n(z) \) may be defined for positive integer \( n \) by the integral

\[ Q_n(z) = \frac{1}{2} \int_{-1}^{1} \frac{(1 - t^2)^n}{2^n (z - t)^{n+1}} dt, \quad z \notin [-1, 1]. \]
Show that for \( x \in [-1, 1] \) we have

\[
Q_n(x + i\epsilon) - Q_n(x - i\epsilon) = -i\pi P_n(x),
\]

where \( P_n(x) \) is the Legendre Polynomial. Deduce Neumann’s formula

\[
Q_n(z) = \frac{1}{2} \int_{-1}^{1} \frac{P_n(t)}{z - t} dt, \quad z \notin [-1, 1].
\]

**Q4 Hilbert transforms:** Suppose that \( \varphi_1(x) \) and \( \varphi_2(x) \) are real functions with finite \( L^2(\mathbb{R}) \) norms.

a) Use the Fourier transform result

\[
\mathcal{F}(\mathcal{H}f)(\omega) = i \text{sgn}(\omega) \mathcal{F}(f)(\omega).
\]

to show that

\[
\langle \varphi_1 | \varphi_2 \rangle = \langle \mathcal{H}\varphi_1 | \mathcal{H}\varphi_2 \rangle.
\]

Thus, \( \mathcal{H} \) is a unitary transformation from \( L^2(\mathbb{R}) \to L^2(\mathbb{R}) \).

b) Use the fact that \( \mathcal{H}^2 = -I \) to deduce that

\[
\langle \mathcal{H}\varphi_1 | \varphi_2 \rangle = -\langle \varphi_1 | \mathcal{H}\varphi_2 \rangle
\]

and so \( \mathcal{H}^* = -\mathcal{H} \).

c) Conclude from part b) that

\[
\int_{-\infty}^{\infty} \varphi_1(x) \left( P \int_{-\infty}^{\infty} \frac{\varphi_2(y)}{x - y} dy \right) dx = \int_{-\infty}^{\infty} \varphi_2(y) \left( P \int_{-\infty}^{\infty} \frac{\varphi_1(x)}{x - y} dx \right) dy,
\]

i.e., for \( L^2(\mathbb{R}) \), functions, it is legitimate to interchange the order of “\( P \)” integration with ordinary integration.

d) By replacing \( \varphi_1(x) \) by a constant, and \( \varphi_2(x) \) by the Hilbert transform of a function \( f \) with \( \int f \, dx \neq 0 \), show that it is not always safe to interchange the order of “\( P \)” integration with ordinary integration.

**Q5 Advanced Hilbert transforms:**

Suppose that are given real functions \( u_1(x) \) and \( u_2(x) \) and substitute their Hilbert transforms \( v_1 = \mathcal{H}u_1, v_2 = \mathcal{H}u_2 \) into (9.78) to construct analytic functions \( f_1(z) \) and \( f_2(z) \). Then the product \( f_1(z)f_2(z) = F(z) \) has boundary value

\[
F_R(x) + iF_I(x) = (u_1u_2 - v_1v_2) + i(u_1v_2 + u_2v_1).
\]

a) By assuming that \( F(z) \) satisfies the conditions for (9.77) to be applicable to this boundary value, deduce that

\[
\mathcal{H}((\mathcal{H}u_1)u_2) + \mathcal{H}((\mathcal{H}u_2)u_1) - (\mathcal{H}u_1)(\mathcal{H}u_2) = -u_1u_2. \quad (*)
\]
This result\(^1\) of part (a) sometimes appears in the physics literature\(^2\) in the guise of the distributional identity

\[
\frac{P}{x - y} \frac{P}{y - z} + \frac{P}{y - z} \frac{P}{z - x} + \frac{P}{z - x} \frac{P}{x - y} = -\pi^2 \delta(x - y) \delta(x - z),
\]

where \(P/(x - y)\) denotes the principal-part distribution \(P\left(\frac{1}{x - y}\right)\). This attractively symmetric form conceals the fact that \(x\) is being kept fixed, while \(y\) and \(z\) are being integrated over in a specific order. As the next part shows, were we to freely re-arrange the integration order we could use the identity

\[
\frac{1}{x - y} \frac{1}{y - z} + \frac{1}{y - z} \frac{1}{z - x} + \frac{1}{z - x} \frac{1}{x - y} = 0 \quad x, y, z \text{ distinct}
\]

to wrongly conclude that the right-hand side is zero.

b) Show that the identity (\(\star\)) can be written as

\[
\int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \frac{\varphi_1(y)\varphi_2(z)}{(z - y)(y - x)} \, dz \right) \, dy = \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \frac{\varphi_1(y)\varphi_2(z)}{(z - y)(y - x)} \, dy \right) \, dz - \pi^2 \varphi_1(x)\varphi_2(x),
\]

principal-part integrals being understood where necessary. This is a special case of a more general change-of-integration-order formula

\[
\int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \frac{f(x, y, z)}{(z - y)(y - x)} \, dz \right) \, dy = \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \frac{f(x, y, z)}{(z - y)(y - x)} \, dy \right) \, dz - \pi^2 f(x, x, x),
\]

which is due to G. H. Hardy (1908). It is usually called the Poincaré-Bertrand theorem.

\(^1\)F. G. Tricomi, *Quart. J. Math. (Oxford)*, (2) 2, (1951) 199.