1 Lobachevski Space

Consider the hyperbolic surface

\[
M = \{ (x, y, z) | z^2 - x^2 - y^2 = R^2, z \geq 0 \}
\]

We can view \( M \) as an embedded manifold in \( \mathbb{R}^3 \), where \( \mathbb{R}^3 \) is endowed with the Minkowski metric \( g = dx \otimes dx + dy \otimes dy - dz \otimes dz \). This induces a metric \( g^{(M)} \) on \( M \).

We can draw an analogy to the metric of \( S^2 \) in regular Euclidean space. Define the "hyperbolic" polar coordinates:

\[
\begin{align*}
x &= R \sinh (\theta) \cos (\phi) \\
y &= R \sinh (\theta) \sin (\phi) \\
z &= R \cosh (\theta)
\end{align*}
\]

Then \( x, y, z \) satisfies \( z^2 - x^2 - y^2 = R^2 \). As such, the induced metric on \( M \) in hyperbolic polars is:

\[
g^M = R^2 d\theta \otimes d\theta + R^2 (\sinh (\theta))^2 d\phi \otimes d\phi
\]

Now, we can translate \( \theta, \phi \) into stereographic coordinates as:

\[
\begin{align*}
\xi &= R \tan (\frac{\theta}{2}) \cos (\phi) \\
\psi &= R \tan (\frac{\theta}{2}) \sin (\phi)
\end{align*}
\]

Under this change of coordinates, the metric \( g^M \) transforms as:

\[
g^M_{\mu \nu} (\theta, \phi) = g^M_{\mu \nu} (\xi, \psi) \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu}
\]

where \((x^1, x^2) = (\theta, \phi)\) and \((y^1, y^2) = (\xi, \psi)\). Define the Jacobian matrix as \( J^\mu_{\nu} = \frac{\partial y^\mu}{\partial x^\nu} \). This gives us the matrix equation:

\[
g = J^t \cdot \tilde{g} \cdot J
\]

where \( J \) is evaluated as:

\[
J = \begin{bmatrix}
\frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \\
\frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi}
\end{bmatrix} = \begin{bmatrix}
\frac{1}{2} R \frac{\cosh (\theta)}{\sinh (\theta/2)} & -R \tanh (\theta/2) \sin (\phi) \\
\frac{1}{2} R \frac{\sinh (\theta)}{\cosh (\theta/2)} & -R \tanh (\theta/2) \cos (\phi)
\end{bmatrix}
\]

Then this gives us:

\[
\tilde{g} = J^{-t} \cdot g \cdot J^{-1} = \begin{bmatrix}
R^2 & 0 \\
0 & R^2 (\sinh (\theta))^2
\end{bmatrix} \cdot J^{-1} = \begin{bmatrix}
4 (\cosh (\theta/2))^4 & 0 \\
0 & 4 (\cosh (\theta/2))^4
\end{bmatrix}
\]

Finally, noting that we can write:

\[
4 (\cosh (\theta/2))^4 = \frac{4R^4}{(R^2 (\sech (\frac{\theta}{2}))^2)^2} = \frac{4R^4}{(R^2 - R^2 (\tanh (\frac{\theta}{2}))^2)^2} = \frac{4R^4}{(R^2 - \xi^2 - \psi^2)^2}
\]

This gives us:

\[
\tilde{g}^M (\xi, \psi) = \frac{4R^4}{(R^2 - \xi^2 - \psi^2)^2} (d\xi \otimes d\xi + d\psi \otimes d\psi)
\]

2 Flywheel + Rolling Ball

Part (a)

In general, the rotational kinetic energy of an extended body is given by:

\[
T = \frac{1}{2} I_{ij} \omega^i \omega_j
\]

where \( I_{ij} \) is the moment of inertia tensor. If we use the body-axis coordinates of the rotating body, we can further simplify the kinetic energy term as:

\[
T = \frac{1}{2} I_x \omega_x^2 + \frac{1}{2} I_y \omega_y^2 + \frac{1}{2} I_z \omega_z^2
\]

Finally, in our case our object exhibits axial symmetry, and we can further simplify that \( I_x = I_y \equiv I_1 \) and \( I_z \equiv I_2 \). This gives our kinetic energy as:

\[
T = \frac{1}{2} I_1 (\omega_x^2 + \omega_y^2) + \frac{1}{2} I_2 \omega_z^2
\]

Using Euler angles (for \( \omega \) w.r.t. to the disk's body axis, in the \( Y \) convention), we have that:

\[
\begin{align*}
\omega_x &= -\dot{\phi} \sin (\theta) \cos (\psi) + \dot{\theta} \sin (\psi) \\
\omega_y &= \dot{\phi} \sin (\theta) \sin (\psi) + \dot{\theta} \cos (\psi) \\
\omega_z &= \dot{\phi} \cos (\theta) + \dot{\psi}
\end{align*}
\]

Using this, and the fact that our object is not subject to an external potential, we obtain our Lagrangian as:

\[
L = \frac{1}{2} I (\dot{\phi} + \dot{\phi} \cos (\theta))^2 + \frac{1}{2} I_2 (\dot{\theta}^2 + (\sin (\theta))^2 \dot{\psi}^2)
\]
From this, if we take the equation of motion w.r.t. to the \( \psi \) variable, we obtain that:

\[
\frac{\partial L}{\partial \psi} = I \left( \dot{\psi} + \dot{\phi} \cos(\theta) \right) \equiv L_Z
\]

\[
\frac{\partial L}{\partial \dot{\psi}} = 0
\]

from which we get that:

\[
\frac{d}{dt} \frac{\partial L}{\partial \psi} = \frac{d}{dt} L_Z = 0
\]

Suppose the axle of rotation undergoes rotational motion; this induces a rotation of the disk about the axle. We can parametrize the motion of the disk about the axle by \( \psi = \psi(t) \), and that of motion of the axle itself by \( \theta = \theta(t) \) and \( \phi = \phi(t) \).

For \( \frac{d}{dt} L_Z = 0 \), then \( L_Z = \dot{\phi} \cos \theta + \dot{\psi} \) is a constant of motion. Suppose now that both the axle and the disk start out at rest; this means that \( \theta(0) = 0 \) and \( \psi(0) = 0 \). As such, we have \( L_Z \big|_{t=0} = 0 \).

By constancy, we have that \( L_Z = 0 \) \( \forall t \). In other words, we have that:

\[
\dot{\psi} = -\dot{\phi} \cos \theta
\]

So, if the axle eventually comes to rest at \( t = \tau \), then \( \dot{\phi} \big|_{\tau} = 0 \), implying \( \dot{\psi} \big|_{\tau} = 0 \). Therefore, the disk also comes to a stop.

Suppose that the tip of the axle traces out a closed curve \( \Gamma = \partial \Omega \) on the surface \( S^2 \). We wish to calculate the change in the rotation of the disk, given by \( \Delta \psi = \int_0^\tau d\psi = \int_0^\tau \dot{\psi} dt \). But this is:

\[
\Delta \psi = \int_0^\tau \frac{d\psi}{dt} dt = -\int_0^\tau \frac{d\psi}{dt} \cos \theta(t) dt
\]

\[
= -\int_t \frac{d\psi}{dt} \cos \theta(\phi) \text{ (viewing } \theta \text{ as parametrized by } \phi \text{ on } \Gamma)\]

\[
= -\int_t \omega \text{ (defining } \omega \equiv d\phi \cos \theta)\]

\[
= -\int_\Omega \omega \text{ (Stokes' Theorem)}
\]

Consider now that \( d\omega \) is explicitly calculated as:

\[
d\omega = (\cos \theta d\phi - \sin \theta d\theta \wedge d\phi = -d(\text{area 2-form on } S^2)
\]

As such we have that:

\[
\Delta \psi = \int_\Omega d\omega = \text{Area}(\Omega)
\]

Alternatively, we can use the area of the complementary region to calculate \( \Delta \psi \). The calculation is similar, except that we arrive at:

\[
\Delta \psi^{(\text{comp})} = \int_{-\Gamma} \omega = -\int_{\Gamma} \omega
\]

\[
= -\int_{S^2-\Omega} d\omega
\]

where the notation \( S^2 - \Omega \) is used to denote the complementary region, enclosed by taking \( \Gamma \) as a boundary, but with opposite orientations. Proceeding, this gives us:

\[
\Delta \psi^{(\text{comp})} = -(4\pi - \text{Area}(\Omega)) = \text{Area}(\Omega) - 4\pi
\]

Note that it would seem that \( \Delta \psi^{(\text{comp})} \neq \Delta \psi \). However, since we cannot directly observe the rotation of the disk at intermediate angles, we can only distinguish the two cases by their final angle. As such, we only care about the angle \( \Delta \psi \) modulo \( 2\pi \); in this framework, we see that \( \Delta \psi^{(\text{comp})} \equiv 2\pi \Delta \psi \), and so are physically indistinguishable.

### Part (b)

Consider the analogous problem of the unit sphere rotating, without slipping, on a flat surface. Using again Euler angles (this time for \( \omega \) w.r.t. to the space axis), we get that the condition for no-slip is:

\[
\dot{\phi} + \psi \cos \theta = 0
\]

which we derived in problem set 2.

Suppose the sphere rolls without slipping on the flat surface s.t. it traces out a closed loop on the surface of the sphere. If we label the point of contact between the sphere and the surface by a vector unit vector \( \mathbf{n} \), \( \mathbf{n} \) starts out point straight down; as the sphere rolls, \( \mathbf{n} \) could change directions, but at the end if the rolling motion traces out a closed curve on the sphere, then \( \mathbf{n} \) must return to its original orientation, which is straight down. As such, the only change in the orientation of the sphere must come in through its rotation about the vertical axis, denoted by \( \phi \).

We wish to calculate \( \Delta \phi \). This is given, in a manner similar to part (a), by:

\[
\Delta \phi = \int_0^\tau d\phi = -\int_0^\tau d\psi \cos \theta
\]

\[
= -\int_{\Gamma} \cos \theta d\psi
\]

\[
= \int_\Omega \sin \theta d\theta \wedge d\psi = \text{Area}(\Omega)
\]

### 3 Hopf Invariant

#### Part (a)

Consider that Euler’s equation for fluid motion is given by:

\[
\frac{Dv}{Dt} \equiv \frac{\partial v}{\partial t} + (v \cdot \nabla) v = -\nabla P
\]

\[
0 = \nabla \cdot v
\]

where \( D/Dt \) is the convective derivative.

#### Part (i)

First we express this equation in components:

\[
\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + v_i \frac{\partial v}{\partial x^i} = \frac{-\partial P}{\partial x^i}
\]

If we now hit the equation with the cross product \( \varepsilon_{a j i} \frac{\partial}{\partial x^j} \), we get:

\[
\varepsilon_{a j i} \frac{\partial}{\partial x^j} \left( v_i \frac{\partial v^j}{\partial x^k} \right) = -\varepsilon_{a j k} \frac{\partial}{\partial x^j} \frac{\partial P}{\partial x^k} \tag{1}
\]
Noting that:

\[ \varepsilon_{a ji} \frac{\partial}{\partial x^j} \frac{\partial P}{\partial x^i} = 0 \]  
contracting of anti-sym with sym

\[ \varepsilon_{a ji} \frac{\partial}{\partial x^j} \frac{\partial v^i}{\partial t} = \frac{\partial}{\partial t} \left( \varepsilon_{a ji} \frac{\partial v^i}{\partial x^j} \right) \equiv \frac{\partial}{\partial t} (\omega^a) \]

\[ \varepsilon_{a ji} \frac{\partial}{\partial x^j} \left( v^k \frac{\partial v^i}{\partial x^k} \right) = \varepsilon_{a ji} \partial_{x^j} \left( v^k \frac{\partial v^i}{\partial x^k} + \varepsilon_{a ji} v^k \frac{\partial v^i}{\partial x^k} \right) \]

\[ = \varepsilon_{a ji} \partial_{x^j} \varepsilon_{a ji} \frac{\partial v^i}{\partial x^k} + v^k \partial_{x^k} \left( \varepsilon_{a ji} \frac{\partial v^i}{\partial x^j} \right) \]

\[ = \varepsilon_{a ji} \partial_{x^j} v^k \frac{\partial v^i}{\partial x^k} + v^k \partial_{x^k} \left( \varepsilon_{a ji} \frac{\partial v^i}{\partial x^j} \right) \]

then equation 1 becomes:

\[ \frac{\partial}{\partial t} (\omega^a) + v^k \frac{\partial}{\partial x^k} (\omega^a) + \varepsilon_{a ji} \frac{\partial v^i}{\partial x^k} \frac{\partial v^j}{\partial x^k} = 0 \]

Consider finally the term \( \varepsilon_{a ji} \partial_{x^j} \frac{\partial v^i}{\partial x^k} \). The claim is that, for \( \nabla \cdot \mathbf{v} = 0 \), we have that:

\[ \varepsilon_{a ji} \partial_{x^j} \frac{\partial v^i}{\partial x^k} = -\omega^a \frac{\partial v^i}{\partial x^k} \]

That this is true can be seen as follows. Consider first that for \( \omega^i = \varepsilon_{ijk} \frac{\partial v^j}{\partial x^k} \), we have that:

\[ \varepsilon_{i lm} \omega^l = \varepsilon_{i lm} \varepsilon_{ijk} \frac{\partial v^j}{\partial x^k} \]

\[ = \frac{\partial v^m}{\partial x^i} - \frac{\partial v^l}{\partial x^i} \]

this gives us that:

\[ \frac{\partial v^l}{\partial x^i} = \varepsilon_{sk l i} \omega^s + \frac{\partial v^l}{\partial x^m} \]

as such, we have:

\[ \varepsilon_{a ji} \partial_{x^j} \frac{\partial v^i}{\partial x^k} = \varepsilon_{a ji} \partial_{x^j} \left( \varepsilon_{sk i} \omega^s + \frac{\partial v^l}{\partial x^m} \right) \]

\[ = \frac{\partial v^l}{\partial x^m} \omega^s - \omega^s \frac{\partial v^l}{\partial x^m} + \varepsilon_{a ji} \frac{\partial v^l}{\partial x^m} \frac{\partial v^i}{\partial x^k} \]

but again we can make simplificaitons:

\[ \frac{\partial v^l}{\partial x^m} \omega^s = 0 \quad \text{using \( \nabla \cdot \mathbf{v} = 0 \)} \]

\[ \varepsilon_{a ji} \frac{\partial v^l}{\partial x^m} = 0 \quad \text{contracting anti-sym with sym} \]

So we have just that:

\[ \varepsilon_{a ji} \frac{\partial v^l}{\partial x^m} \frac{\partial v^i}{\partial x^k} = -\omega^s \frac{\partial v^l}{\partial x^m} \]

As such, our final form of Euler’s equations become:

\[ \frac{\partial}{\partial t} (\omega^a) + v^k \frac{\partial}{\partial x^k} (\omega^a) = \omega^a \frac{\partial v^a}{\partial x^a} \]

which is another way of writing that:

\[ \frac{\partial}{\partial t} \omega^a + (\mathbf{v} \cdot \nabla) \omega = (\omega \cdot \nabla) \mathbf{v} \]

or as a convective derivative:

\[ \frac{D}{Dt} (\omega^a) = (\omega \cdot \nabla) \mathbf{v} \]

**Part (ii)**

We wish to show that:

\[ \frac{D}{Dt} (\mathbf{v} \cdot \omega) = \nabla \cdot \left( \omega \left( \frac{1}{2} \mathbf{v}^2 - P \right) \right) \]

which in components is:

\[ \left( \frac{D}{Dt} (v^i \omega^j) \right) = \frac{\partial}{\partial x^j} \left( \omega^s \left( \frac{1}{2} v^i v^j - P \right) \right) \]

our work is simplified if we can show that:

\[ \frac{D}{Dt} (fg) = \frac{D}{Dt} (f) \cdot g + f \cdot \frac{D}{Dt} (g) \]

To this end, consider that:

\[ \frac{D}{Dt} (fg) = \frac{\partial}{\partial t} (fg) + v^s \frac{\partial}{\partial x^s} (fg) \]

\[ = \frac{\partial f}{\partial t} g + v^s \frac{\partial f}{\partial x^s} g + f \frac{\partial g}{\partial t} + f v^s \frac{\partial g}{\partial x^s} \]

\[ = Df \frac{D}{Dt} g + f \frac{D}{Dt} g \]

As such, we have that:

\[ \frac{D}{Dt} (v^i \omega^j) = \frac{D}{Dt} (v^i) \omega^j + v^s \frac{D}{Dt} (\omega^j) \]

Citing Euler’s equation and the results from part (a), we have that:

\[ \frac{D}{Dt} (v^i \omega^j) = -\omega^s \frac{\partial P}{\partial x^s} + \omega^j v^s \frac{\partial v^i}{\partial x^s} \]

Consider now the RHS of equation 2, which is:

\[ \frac{\partial}{\partial x^s} \left( \omega^s \left( \frac{1}{2} v^i v^j - P \right) \right) = \omega^s \frac{\partial}{\partial x^s} \left( \frac{1}{2} v^i v^j - P \right) + \omega^j v^s \frac{\partial v^i}{\partial x^s} \]

However, since \( \omega^s = \varepsilon_{sk l i} \frac{\partial v^j}{\partial x^k} \) then \( \frac{\partial}{\partial x^s} \omega^s = 0 \). As such, we just have:

\[ \frac{\partial}{\partial x^s} \left( \omega^s \left( \frac{1}{2} v^i v^j - P \right) \right) = -\omega^s \frac{\partial P}{\partial x^s} + v^s \frac{\partial v^i}{\partial x^s} \]

so the LHS are equivalent, and so we have:

\[ \left( \frac{D}{Dt} (v^i \omega^j) \right) = \frac{\partial}{\partial x^s} \left( \omega^s \left( \frac{1}{2} v^i v^j - P \right) \right) \]

**Part (iii)**

Consider the integral:

\[ I(t) = \int_{\Omega(t)} f(x,t) \, d^n x \]

where \( f(x,t) \) is a function defined on a region \( \Omega(t) \subset \mathbb{R}^n \).

We wish to show that, for a region \( \Omega(t) \subset \mathbb{R}^n \) that is also comoving with the incompressible fluid, then we should have:

\[ \frac{d}{dt} \int_{\Omega(t)} f(x,t) \, d^n x = \int_{\Omega(t)} \frac{Df}{Dt} \, d^n x \]

In general, for fixed coordinates \( x \) (coordinates which do not change w.r.t. \( t \)), we can define “comoving coordinates” \( u \) as follows:
Define $u = u(x,t)$, where $u$ is a smooth, invertible mapping $u : \Omega(t) \to M$ to some fixed space $M \subset \mathbb{R}^n$, so that:

$$\frac{du}{dt}igg|_p = 0 \quad \forall p \in M, \quad \forall t$$

$$u(q,t) \in \partial M \quad \forall q \in \partial \Omega(t), \quad \forall t$$

($u$ is essentially a time-varied homeomorphism to $M$; alternatively we can think of $\Omega(t)$ as having been arrived at by some time-varied deformation of $M$). We can imagine the coordinates $u$ as being “glued-down” to the region $\Omega(t)$; so as $\Omega(t)$ changes, the points $(p,t)$ remain constant. The advantage of this is that we can trade our integral time-varying integration region $\int_{\Omega(t)}$ for a time-constant integration region $\int_M$ as follows:

$$\int_{\Omega(t)} f(x,t) \, d^n x = \int_M f(x(u(t),t)) \, J(u,t) \, d^n u$$

where $x = x(u,t)$ is the inverse mapping of $u = u(x,t)$, and $J(u,t)$ is the Jacobian $J(u,t) = \det \left( \frac{\partial x}{\partial u} \right)$. Effectively, all the time-dependence is now on the integrand $f$ and the volume element $J(u,t) \, d^n u$.

As such, we can proceed by taking the total time derivative as follows:

$$\frac{d}{dt} \left( \int_{\Omega(t)} f(x,t) \, d^n x \right) = \frac{d}{dt} \int_{\Omega(t)} f(x,u(t),t) \, J(u,t) \, d^n u$$

$$= \int_M \frac{d}{dt} \left( f(x(u(t),t)) \right) J(u,t) \, d^n u$$

$$= \int_M \frac{d}{dt} J(u,t) \, d^n u + \int_M J(u,t) \, \frac{d}{dt} d^n u$$

$$= \int_M \left( \frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial x^j} \frac{dx^j}{dt} \right) J(u,t) \, d^n u + \int_M J(u,t) \, \frac{d}{dt} d^n u$$

$$= \int_M \frac{Df}{Dt} J(u,t) \, d^n u + \int_M J(u,t) \, \frac{d}{dt} d^n u$$

For a general mapping $u$, this would be the final result. However, if we have the condition that $\Omega(t)$ is an **incompressible** fluid ($\nabla \cdot v = \frac{\partial x^i}{\partial u^j} \frac{dx^j}{dt} = 0$), then $\frac{d}{dt} = 0$. That is true can be seen as follows.

First, we cite a useful determinant identity (Appendix A, approximately eqn A.85 depending on version, of the semester I lecture notes): For an invertible matrix $A$, we have that:

$$\frac{d}{dt} \left( \det A \right) = \text{Tr} \left( \frac{dA}{dt} A^{-1} \right)$$

In particular, for a Jacobian matrix $J$, we have that $(J)^i_j = \frac{\partial x^i}{\partial u^j}$. However, for an invertible mapping $u = u(x,t) \leftrightarrow x = x(u,t)$, we must have that:

- $J \equiv \det J \neq 0$
- $(J^{-1})^i_j = \frac{\partial u^i}{\partial x^j}$

As such, the above identity reduces to:

$$\frac{d}{dt} \left( \det J \right) = \text{Tr} \left( \frac{dJ}{dt} J^{-1} \right) = \frac{d}{dt} \left( \frac{\partial x^i}{\partial u^j} \right) \frac{\partial u^j}{\partial x^i}$$

$$= \frac{\partial}{\partial x^i} \left( \frac{dx^i}{dt} \right) \frac{\partial u^j}{\partial x^i}$$

$$= \frac{\partial}{\partial x^i} \left( \frac{dx^i}{dt} \right) = \nabla \cdot \mathbf{v}$$

So for an incompressible fluid, we have that $\frac{\partial}{\partial x^i} \left( \frac{dx^i}{dt} \right) = 0$. This gives us that $\frac{d}{dt} \left( \frac{d^*}{dt} \right) = 0$, which for an invertible map $J \neq 0$ leads to:

$$\frac{d}{dt} \left( \frac{d^*}{dt} \right) = 0$$

Alternatively, we could’ve derived this result if we can show that the condition $\nabla \cdot \mathbf{v} = 0$ leads to $\text{vol}(\Omega(t)) = \int_{\Omega(t)} d^* x = \text{const}$, in which case we can simply show that:

$$\frac{d}{dt} \text{vol}(\Omega(t)) = \int_{\Omega(t)} d^n x = \int_{\Omega(t)} J(u,t) \, d^n u$$

and since this result holds for arbitrary volumes $\Omega(t)$, then it must be that $\frac{d}{dt} = 0$.

In any event, having shown that $\frac{d}{dt} = 0$, then $\int_M \frac{d}{dt} d^n u = 0$ in equation 3, which leaves us with:

$$\frac{d}{dt} = \int_M \frac{Df}{Dt} J(u,t) \, d^n u$$

$$= \int_{\Omega(t)} \frac{Df}{Dt} J(u,t) \, d^n u$$

as desired.

**Part (iv)**

Having shown that $\frac{d}{dt} \int_\Omega f(x,t) \, dV = \int_\Omega \frac{Df}{Dt} dV$, then applying it to the case that $f = \mathbf{v} \cdot \omega$ gives us:

$$\frac{d}{dt} \int_\Omega f \cdot \omega dV = \int_\Omega \frac{D}{Dt} \left( \mathbf{v} \cdot \omega \right) dV$$

Defining $H = \int_\Omega \mathbf{v} \cdot \omega dV$ and using part (ii), this gives us that:

$$\frac{d}{dt} \int_\Omega \mathbf{v} \cdot \omega dV = \frac{d}{dt} \int_\Omega \mathbf{v} \cdot \left( \mathbf{v} \cdot \frac{\omega}{2} \mathbf{v} - P \right) dV$$

$$= \frac{d}{dt} \int_\Omega \omega \left( \frac{1}{2} \mathbf{v}^2 - P \right) \cdot dS$$

where $\Omega$ is all of space. As such, if $\omega$ vanishes at infinity, this means that $\omega|_{\partial \Omega} = 0$ so the integral vanishes, giving us $\frac{d}{dt} H = 0$.

**Part (b)**

Consider that the electromagnetic field is expressed in terms of the potential $(\mathbf{A}, \varphi)$ as:

$$B^i = \varepsilon_{ijk} \frac{\partial A^k}{\partial x^j}$$

$$E^i = -\frac{\partial A^i}{\partial t} - \frac{\partial \varphi}{\partial x^i}$$

or:

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \varphi$$

In an incompressible fluid, we must have that $E^i + \varepsilon_{ijk} v^j B^k = 0$, or $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$
Part (i)

We wish to show that the above equations imply:
\[
\frac{\partial A}{\partial t} = \mathbf{v} \times (\nabla \times A) - \nabla \phi
\]
\[
\frac{\partial B}{\partial t} = \nabla \times (\mathbf{v} \times B)
\]
Consider that we have:
\[
\frac{\partial A}{\partial t} = -\mathbf{E} - \nabla \phi
\]
\[
= \mathbf{v} \times \mathbf{B} - \nabla \phi
\]
\[
= \mathbf{v} \times (\nabla \times A) - \nabla \phi
\]
and we also have:
\[
\frac{\partial B}{\partial t} = \frac{\partial}{\partial t} (\nabla \times A) = \nabla \times \frac{\partial A}{\partial t}
\]
\[
= \nabla \times (-\mathbf{E} - \nabla \phi)
\]
\[
= -\nabla \times \mathbf{E} \quad (\nabla \times \nabla \phi = 0)
\]
\[
= \nabla \times (\mathbf{v} \times B)
\]

Part (ii)

We wish to show that:
\[
\frac{D}{Dt} (\mathbf{A} \cdot \mathbf{B}) = \nabla \cdot (\mathbf{B} (\mathbf{A} \cdot \mathbf{v} - \phi))
\]
Using the fact that the convective derivative satisfies the product rule, we have that:
\[
\frac{D}{Dt} (\mathbf{A} \cdot \mathbf{B}) = \frac{D}{Dt} (\mathbf{A} \cdot \mathbf{B}) + \mathbf{A} \cdot \frac{D}{Dt} (\mathbf{B})
\]
or that:
\[
\frac{D}{Dt} (A^k B^k) = \frac{D}{Dt} (B^k) A^k + \frac{D}{Dt} (A^k) B^k
\]
Consider firstly that we have:
\[
A^k \frac{D}{Dt} (B^k) = \frac{D}{Dt} (A^k B^k) = A^k \left( \frac{\partial B^k}{\partial t} + \mathbf{v}^a \partial_a B^k \right)
\]
\[
= \mathbf{A}^k \left( \varepsilon_{kab} \partial_a (\varepsilon_{bij} \mathbf{v}^j B^i) + \mathbf{v}^a \partial_a B^k \right)
\]
\[
= \mathbf{A}^k \varepsilon_{kab} \varepsilon_{bij} \partial_a (\mathbf{v}^j B^i) + \mathbf{v}^a \mathbf{A}^k \left( \partial_a B^k \right)
\]
\[
= \mathbf{A}^k \left( \partial_a (\mathbf{v}^k B^a) - \partial_a (\mathbf{v}_a B^k) \right) + \mathbf{v}^a \mathbf{A}^k \left( \partial_a B^k \right)
\]
\[
= \left( \partial_a \mathbf{v}^k \right) A^k B^a - \mathbf{v}^a B^k \left( \partial_a A^k \right) + \mathbf{v}^a A^k \left( \partial_a B^k \right)
\]
\[
= \left( \partial_a \mathbf{v}^k \right) A^k B^a - \mathbf{v}^a B^k \left( \partial_a A^k \right) + \mathbf{v}^a A^k \left( \partial_a B^k \right)
\]
\[
= \left( \partial_a \mathbf{v}^k \right) A^k B^a - \mathbf{v}^a B^k \left( \partial_a A^k \right) + \mathbf{v}^a A^k \left( \partial_a B^k \right)
\]
\[
\text{using } \nabla \cdot \mathbf{v} = 0 \text{ and } \nabla \cdot \mathbf{B} = 0
\]
\[
= \left( \partial_a \mathbf{v}^k \right) A^k B^a
\]
and secondly that we have:
\[
\frac{D}{Dt} (A^k) B^k = \left( \frac{\partial}{\partial t} A^k + \mathbf{v}^a \partial_a A^k \right) B^k
\]
\[
= \varepsilon_{kab} \varepsilon_{bij} \left( \partial_a \mathbf{A}^j \right) B^k - \left( \partial_a \phi \right) B^k + \mathbf{v}^a \left( \partial_a A^k \right) B^k
\]
\[
= \mathbf{v}^a \left( \partial_a A^k \right) B^k - \left( \partial_a \phi \right) B^k
\]
This gives us that:
\[
\frac{D}{Dt} (A^k B^k) = \left( \partial_a \mathbf{v}^k \right) A^k B^a + \mathbf{v}^a \left( \partial_a A^k \right) B^k - \left( \partial_a \phi \right) B^k
\]
Consider now the term:
\[
\text{RHS} = \nabla \cdot (\mathbf{B} (\mathbf{A} \cdot \mathbf{v} - \phi))
\]
\[
= \partial_k \left( B^k (A^a v^a - \phi) \right)
\]
\[
= B^k \partial_k (A^a v^a - \phi) \quad \text{using } \nabla \cdot \mathbf{B} = 0
\]
\[
= v^a (\partial_k A^a) B^k + (\partial_k v^a) A^k B^k - (\partial_k \phi) B^k
\]
\[
= \frac{D}{Dt} (A^k B^k)
\]
Therefore we have that LHS = RHS.

Part (iii)

Consider now if we define:
\[
W = \int_{\Omega} (\mathbf{A} \cdot \mathbf{B}) \, dV
\]
then by the same reasoning as in part (a), we will have that:
\[
\frac{d}{dt} W = \frac{d}{dt} \int_{\Omega} (\mathbf{A} \cdot \mathbf{B}) \, dV
\]
\[
= \int_{\Omega} \frac{D}{Dt} (\mathbf{A} \cdot \mathbf{B}) \, dV
\]
\[
= \int_{\Omega} \nabla \cdot (\mathbf{B} (\mathbf{A} \cdot \mathbf{v} - \phi)) \, dV
\]
\[
= \int_{\partial \Omega} \mathbf{B} (\mathbf{A} \cdot \mathbf{v} - \phi) \cdot dS
\]
and so if B = 0 at infinity, then the integral vanishes, and so W is a constant.

4 Faraday’s Law

Consider the Faraday tensor \( F = \frac{1}{2} F_{\mu \nu} dx^\mu \wedge dx^\nu \), defined on 2D time-evolving manifold \( \Omega(\tau) \subset M^2 \), viewed as an embedded manifold in Minkowski space. Let \( (\eta^1, \eta^2) = (u, v) \) be local, co-moving coordinates on \( \Omega \), and \( x^a \) be local coordinates for Minkowski space.

Part (a)

Consider the integral:
\[
I(\tau) = \int_{\Omega(\tau)} F (x) = \int_{\Omega(\tau)} \frac{1}{2} F_{\mu \nu} (x) \, dx^\mu \wedge dx^\nu
\]
Similar to question 3, since \( u, v \) are co-moving coordinates, we can reparametrize the coordinates \( x = x(\eta, \tau) \), the integration region \( \Omega(\tau) \to \tilde{\Omega} \) (where \( \tilde{\Omega} \) is \( \tau \)-independent), and the integral as:
\[
\int_{\tilde{\Omega}} F (x(\eta, \tau)) = \int_{\tilde{\Omega}} \frac{1}{2} F_{\mu \nu} (x(\eta, \tau)) \, dx^\mu \wedge dx^\nu
\]
\[
= \int_{\eta^2} \frac{1}{2} F_{\mu \nu} (x(\eta, \tau)) \, dx^\mu \wedge dx^\nu
\]
\[
= \int_{\eta^2} \frac{1}{2} F_{\mu \nu} (x(\eta, \tau)) \, dx^\mu \wedge dx^\nu
\]
(note that $\frac{\partial x^\mu}{\partial \eta^\alpha} \frac{\partial x^\nu}{\partial \eta^\beta}$ is analogous to the Jacobian term in question 3, except in that case the time derivative will not vanish since we are not considering an incompressible flow). Proceeding now to take the time derivative:

$$\frac{d}{d\tau} I(\tau) = \frac{d}{d\tau} \int_\Omega \frac{1}{2} F_{\mu\nu}(x(\eta, \tau)) \left( \frac{\partial x^\mu}{\partial \eta^\alpha} \frac{\partial x^\nu}{\partial \eta^\beta} \right) d\eta^\alpha d\eta^\beta$$

$$= \int_\Omega \frac{1}{2} \frac{d}{d\tau} \left( F_{\mu\nu}(x(\eta, \tau)) \right) \left( \frac{\partial x^\mu}{\partial \eta^\alpha} \frac{\partial x^\nu}{\partial \eta^\beta} \right) d\eta^\alpha d\eta^\beta$$

$$+ \int_\Omega \frac{1}{2} F_{\mu\nu}(x(\eta, \tau)) \frac{d}{d\tau} \left( \frac{\partial x^\mu}{\partial \eta^\alpha} \frac{\partial x^\nu}{\partial \eta^\beta} \right) d\eta^\alpha d\eta^\beta$$

$$= \int_\Omega \frac{1}{2} \frac{\partial}{\partial \tau} \left( F_{\mu\nu}(x(\eta, \tau)) \right) \left( \frac{\partial x^\mu}{\partial \eta^\alpha} \frac{\partial x^\nu}{\partial \eta^\beta} \right) d\eta^\alpha d\eta^\beta$$

Now consider the $\frac{d}{d\tau} \left( \frac{\partial x^\mu}{\partial \eta^\alpha} \right)$ term. We have:

$$\frac{d}{d\tau} \left( \frac{\partial x^\mu}{\partial \eta^\alpha} \right) = \frac{\partial}{\partial \tau} \left( \frac{\partial x^\mu}{\partial \eta^\alpha} \right) = \frac{\partial}{\partial \tau} \left( \frac{dx^\mu}{d\tau} \right)$$

and likewise the $\frac{d}{d\tau} \left( \frac{\partial x^\nu}{\partial \eta^\beta} \right)$ term. If we write $V^\mu \equiv \frac{dx^\mu}{d\tau}$, then we can combine the above as:

$$\frac{d}{d\tau} I(\tau) = \frac{1}{2} \int_\Omega \left( \frac{\partial F_{\mu\nu}}{\partial x^\sigma} V^\sigma + F_{\rho\sigma} \frac{\partial V^\nu}{\partial x^\mu} + F_{\mu\sigma} \frac{\partial V^\nu}{\partial x^\rho} \right) \left( \frac{dx^\mu}{d\tau} \right) \left( \frac{dx^\nu}{d\tau} \right) d\eta^\alpha d\eta^\beta$$

$$= \frac{1}{2} \int_\Omega \left( \frac{\partial F_{\mu\nu}}{\partial x^\sigma} V^\sigma + F_{\rho\sigma} \frac{\partial V^\nu}{\partial x^\mu} + F_{\mu\sigma} \frac{\partial V^\nu}{\partial x^\rho} \right) \left( \frac{dx^\mu}{d\tau} \right) \left( \frac{dx^\nu}{d\tau} \right)$$

Recalling that for a general rank 2 tensor $T$, we have that:

$$(\mathcal{L}_X T)_{\mu\nu} = \frac{\partial T_{\mu\nu}}{\partial x^\sigma} X^\sigma + T_{\rho\sigma} \frac{\partial X^\mu}{\partial x^\rho} + T_{\mu\rho} \frac{\partial X^\nu}{\partial x^\rho}$$

this gives us:

$$\frac{d}{d\tau} I(\tau) = \frac{1}{2} \int_{\Omega(\tau)} (\mathcal{L}_V F)_{\mu\nu} \left( \frac{dx^\mu}{d\tau} \right) \left( \frac{dx^\nu}{d\tau} \right)$$

$$= \int_\Omega F \mathcal{L}_V F$$

(4)

Consider furthermore that by the infinitesimal homotopy relation, we have that:

$$\int_{\Omega(\tau)} \mathcal{L}_F V = \int_{\Omega(\tau)} \text{div}_V F = \int_{\partial \Omega(\tau)} i_V dF$$

However, for $F = \mathcal{L}_V F$, then $dF = 0$, and we just have:

$$\int_{\Omega(\tau)} \mathcal{L}_F V = \int_{\Omega(\tau)} i_V F = \int_{\partial \Omega(\tau)} i_V F$$

(5)

by Stokes' theorem. Finally, defining the Lorentz four-force as $f = -i_V F$, and combining equations 4 and 5, we arrive at:

$$\frac{d}{d\tau} \int_{\Omega(\tau)} F = \int_{\Omega(\tau)} \mathcal{L}_V F = \int_{\Omega(\tau)} i_V F = -\int_{\partial \Omega(\tau)} f$$

Part (b)

Suppose the position four-vector is given by $x^\mu = (t, \tau)$. Then for $\tau$ proper time, we have that:

$$\frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt} \frac{dt}{d\tau} = \frac{dx^\mu}{dt} \gamma$$

$$= \frac{1}{\sqrt{1 - \beta^2}} \left( \frac{dx}{dt} \right)$$

where $\beta = |dx/dt|$ (in units of $c = 1$).

Likewise, the Lorentz four-force is given by:

$$f_\mu = (-i_V F)_\mu = -F_{\nu\mu} V^\nu = F_{\nu\mu} V^\nu$$