1) Counting indices: Show that in d dimensions:

i) the dimension of the space of skew-symmetric covariant tensors with p indices is $d!/p!(d-p)!$;

ii) the dimension of the space of symmetric covariant tensors with p indices is $(d+p-1)!/p!(d-1)!$.

2) Quantum Entanglement: Two quantum mechanical systems have Hilbert spaces $H^{(1)}$ with basis $\mathbf{e}_1^{(1)}, \ldots, \mathbf{e}_m^{(1)}$ and $H^{(2)}$ with basis $\mathbf{e}_1^{(2)}, \ldots, \mathbf{e}_n^{(2)}$. The Hilbert space for the combined system is then $H^{(1)} \otimes H^{(2)}$ with basis $\mathbf{e}_i^{(1)} \otimes \mathbf{e}_j^{(2)}$, so the quantum state of the combined system is described by a state

$$\mathbf{a} = a^{ij} \mathbf{e}_i^{(1)} \otimes \mathbf{e}_j^{(2)} \in H^{(1)} \otimes H^{(2)}.$$

If we can find vectors

$$\mathbf{x} = x^i \mathbf{e}_i^{(1)} \in H^{(1)}$$
$$\mathbf{y} = y^j \mathbf{e}_j^{(2)} \in H^{(2)}$$

such that

$$\mathbf{a} = \mathbf{x} \otimes \mathbf{y} = x^i y^j \mathbf{e}_i^{(1)} \otimes \mathbf{e}_j^{(2)}$$

then the tensor $\mathbf{a}$ is said to be *decomposable* and the two quantum systems are said to be *unentangled*. If there are no such vectors the two systems are *entangled* in the sense of the Einstein-Podolski-Rosen (EPR) paradox.

i) By counting the number of components that are at our disposal $\mathbf{a}$ and in $\mathbf{x} \otimes \mathbf{y}$ find out how many relations the coefficients a_{ij} must satisfy if the state is to be decomposable.

ii) If the state is decomposable, show that

$$0 = \begin{vmatrix} a^{ij} & a^{il} \\ a^{kj} & a^{kl} \end{vmatrix}$$

for all sets of indices i, j, k, l.

iii) Using your result from part i) as a reality check, find a subset of the relations from part ii), that constitute a necessary and sufficient set of conditions for the state $\mathbf{a}$ to be decomposable. Include a proof that your set is indeed sufficient.

Since quantum states are really in one-to-one correspondence with *rays* in the Hilbert space, rather than vectors, the set of decomposable states should be thought of as a subset of the complex projective space $\mathbf{CP}^{nm-1}$, and since it is defined by a finite number of polynomial equations it forms what algebraic geometers call a *variety*. This particular subset is known as the *Segre variety*.

3) Symmetric integration: Show that the n-dimensional integral

$$I_{\alpha\beta\gamma\delta} = \int \frac{d^n k}{(2\pi)^n} (k_\alpha k_\beta k_\gamma k_\delta)\, f(k^2),$$

is equal to

$$A(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma})$$

where

$$A = \frac{1}{n(n+2)} \int \frac{d^n k}{(2\pi)^n} (k^2)^2 f(k^2).$$

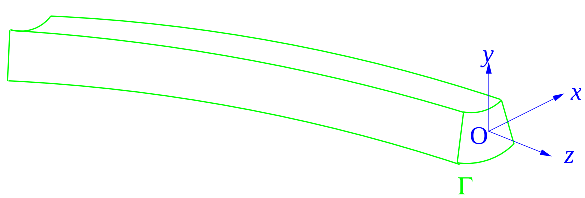
Similarly evaluate

$$I_{\alpha\beta\gamma\delta\epsilon} = \int \frac{d^n k}{(2\pi)^n} (k_\alpha k_\beta k_\gamma k_\delta k_\epsilon)\, f(k^2).$$

4) Leonardo da Vinci's problem II: A steel beam is forged so that its cross section has the shape of a region $\Gamma \in \mathcal{R}^2$. The centroid, O, of each cross section is defined so that

$$\int_\Gamma x\, dxdy = \int_\Gamma y\, dxdy = 0,$$

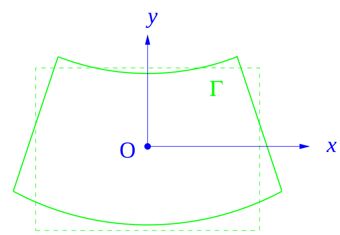
where the co-ordinates x, y are defined with the centroid O as the origin. The beam is slightly bent in the $y - z$ plane so that near a particular cross-section the line of centroids has radius of curvature R. (In the figure this cross section is depicted at the end of the beam. It is actually an interior slice)



Bent beam.

Assume that near O the deformation is such that

$$\begin{aligned}
\eta_x &= -\frac{\sigma}{R} xy \\
\eta_y &= \frac{1}{2R}\left\{\sigma(x^2 - y^2) - z^2\right\} \\
\eta_z &= \frac{1}{R} yz
\end{aligned}$$



The deformed cross-section (greatly exagerated).

Verify that this distortion field does correpond to the beam being bent downwards, and with the line of centroids having radius of curvature R. Notice how, for positive Poisson ratio, the cross section is deformed *anticlastically* — the sides bend *up* as the beam bends *down*. Show that

$$e_{xx} = -\frac{\sigma}{R} y, \qquad e_{yy} = -\frac{\sigma}{R} y, \qquad e_{zz} = \frac{1}{R} y.$$

Since steel is isotropic, the stresses are derived from the strains *via* the Lamé constants. Show that $\sigma_{zz} = Yy/R$, where Y is Young's modulus, and that all other components of the stress tensor vanish. Deduce from this that the assumed deformation satisfies the force-free surface boundary condition, and so is indeed the way the beam deforms. The total elastic energy is given by

$$E = \iiint_{\text{beam}} \frac{1}{2} e_{ij} c_{ijkl} e_{kl} \, d^3x.$$

Show that for our bent beam, this reduces to

$$E = \int \frac{YI}{2} \left(\frac{1}{R^2} \right) ds \approx \int \frac{YI}{2} (y'')^2 dz.$$

Here s is the arc-length taken along the line of centroids of the beam,

$$I = \int_\Gamma y^2 \, dxdy$$

is the moment of inertia of the region Γ about an axis through the centroid, and perpendicular both to the length of the beam and the plane into which it is bent. The right-hand-side formula is the expression used many times in 508. Here y denotes the deflection of the beam away from the z axis, and the primes denote differentiation with respect to z or s.

5) Maxwell Stress: Let

$$\Pi_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2} \delta_{ij} |E|^2 \right) + \mu_0 \left(H_i H_j - \frac{1}{2} \delta_{ij} |H|^2 \right).$$

Show that Maxwell's equations lead to

$$(\rho \mathbf{E} + \mathbf{j} \times \mathbf{B})_i + \frac{\partial}{\partial t} \left\{ \frac{1}{c^2} (\mathbf{E} \times \mathbf{H})_i \right\} = \partial_j \Pi_{ji}.$$