1) Möbius Maps: The Map

$$z \mapsto w = \frac{az+b}{cz+d}$$

is called a Möbius transformation. These maps are important because they are the only one-to-one conformal maps of the Riemann sphere onto itself.

a) Show that two successive Möbius transformations

$$z' = \frac{az+b}{cz+d}, \qquad z'' = \frac{Az'+B}{Cz'+D}$$

give rise to another Möbius transformation, and show that the rule for combining them is equivalent to matrix multiplication.

b) Let z_1, z_2, z_3, z_4 be complex numbers. Show that a necessary and sufficient condition for the four points to be concyclic is that their their *cross-ratio*

$$\{z_1, z_2, z_3, z_4\} \stackrel{\text{def}}{=} \frac{(z_1-z_4)(z_3-z_2)}{(z_1-z_2)(z_3-z_4)}$$

be real (Hint: use a well-known property of opposite angles of a cyclic quadrilateral). Show that Möbius transformations leave the cross-ratio invariant, and thus take circles into circles.

2) Hyperbolic geometry: The Riemann metric for the Poincaré-disc model of Lobachevski's hyperbolic plane can be taken to be

$$ds^2 = \frac{4|dz|^2}{(1-|z|^2)^2}, \quad |z|^2 < 1.$$

a) Show that the Möbius transformation

$$z \mapsto w = e^{i\lambda}\frac{z-a}{\bar{a}z-1}, \qquad |a| < 1, \quad \lambda \in \mathbb{R}$$

provides a 1-1 map of the interior of the unit disc onto itself. Show that these maps form a group.

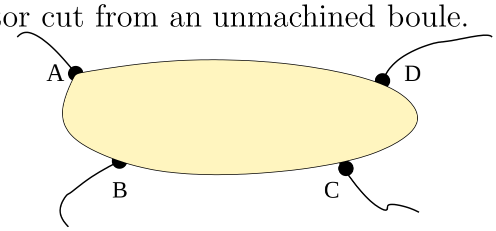
b) Show that the hyperbolic-plane metric is left invariant under the group of maps in part (a). Deduce that such maps are orientation-preserving *isometries* of the hyperbolic plane.

c) Use the circle-preserving property of the Möbius maps to deduce that circles in hyperbolic geometry are represented in the Poincaré disc by Euclidean circles that lie entirely within the disc.

The conformal maps of part (a) are in fact the *only* orientation preserving isometries of the hyperbolic plane. With the exception of circles centered at $z = 0$, the center of the hyperbolic circle does not coincide with the center of its representative Euclidean circle. Euclidean circles that are internally tangent to the boundary of the unit disc have infinite hyperbolic radius and their hyperbolic centers lie on the boundary of the unit disc and hence at hyperbolic infinity. They are known as *horocycles.*

3) Rectangle to Ellipse: Consider the map $w \mapsto z = \sin w$. Draw a picture of the image, in the z plane, of the interior of the rectangle with corners $u = \pm\pi/2$, $v = \pm\lambda$. ($w = u + iv$). Show which points correspond to the corners of the rectangle, and verify that the vertex angles remain $\pi/2$. At what points does the isogonal property fail?

4) Van der Pauw's Theorem: This problem explains a practical method of for determining the conductivity σ of a material, given a sample in the form of of a wafer of uniform thickness d, but of irregular shape. In practice at the Phillips company in Eindhoven, this was a wafer of semiconductor cut from an unmachined boule.



A thin semiconductor wafer with attached leads.

We attach leads to point contacts A, B, C, D, taken in anticlockwise order, on the periphery of the wafer and drive a current I_{AB} from A to B. We record the potential difference $V_D - V_C$ and so find $R_{AB,DC} = (V_D - V_C)/I_{AB}$. Similarly we measure $R_{BC,AD}$. The current flow in the wafer is assumed to be two dimensional, and to obey

$$\mathbf{J} = -(\sigma d)\nabla V, \qquad \nabla \cdot \mathbf{J} = 0,$$

and $\mathbf{n} \cdot \mathbf{J} = 0$ at the boundary (except at the current source and drain). The potential V is therefore harmonic, with Neumann boundary conditions.

Van der Pauw claims that

$$\exp\{-\pi\sigma d R_{AB,DC}\} + \exp\{-\pi\sigma d R_{BC,AD}\} = 1.$$

From this σd can be found numerically.

a) First show that Van der Pauw's claim is true if the wafer were the entire upper half-plane with A, B, C, D on the real axis with $x_A < x_B < x_C < x_D$.

b) Next, taking care to consider the transformation of the current source terms and the Neumann boundary conditions, show that the claim is invariant under conformal maps, and, by mapping the wafer to the upper half-plane, show that it is true in general.

5) Bergman Kernel: The Hilbert space of analytic functions on a domain D with inner product

$$\langle f, g\rangle = \int_D \bar{f} g \, dxdy$$

is called the Bergman space of D.

a) Suppose that $\varphi_n(z)$, $n = 0, 1, 2, \ldots$, are a complete set of orthonormal functions on the Bergman space. Show that

$$K(\zeta, z) = \sum_{m=0}^{\infty} \varphi_m(\zeta)\overline{\varphi_m(z)}.$$

has the property that

$$g(\zeta) = \iint_D K(\zeta, z) g(z) \, dxdy.$$

for any function g analytic in D. Thus $K(\zeta, z)$ plays the role of the delta function on the space of analytic functions on D. This object is called the *reproducing* or *Bergman kernel*. By taking $g(z) = \varphi_n(z)$, show that it is the unique integral kernel with the reproducing property.

b) Consider the case of D being the unit disc. Use the Gramm-Schmidt procedure to construct an orthonormal set from the functions z^n, $n = 0, 1, 2, \ldots$. Use the result of part a) to conjecture (because we have not proved that the set is complete) that, for the unit disc,

$$K(\zeta, z) = \frac{1}{\pi} \frac{1}{(1 - \zeta\bar{z})^2}.$$

c) For any smooth, complex valued, function g defined on a domain D and its boundary, use Stokes' theorem to show that

$$\iint_D \partial_{\overline{z}} g(z, \overline{z}) dxdy = \frac{1}{2i} \oint_{\partial D} g(z, \overline{z}) dz.$$

Use this to verify that this the $K(\zeta, z)$ you constructed in part b) is indeed a (and hence "the") reproducing kernel.

d) Now suppose that D is a simply connected domain whose boundary ∂D is a smooth curve. We know from the Riemann mapping theorem that there exists an analytic function $f(z) = f(z; \zeta)$ that maps D onto the interior of the unit circle in such a way that $f(\zeta) = 0$ and $f'(\zeta)$ is real and non-zero. Show that if we set $K(\zeta, z) = \overline{f'(z)} f'(\zeta)/\pi$, then, by using part c) together with the residue theorem to evaluate the integral over the boundary, we have

$$g(\zeta) = \iint_D K(\zeta, z) g(z) \, dxdy.$$

This $K(\zeta, z)$ must therefore be the reproducing kernel. We see that if we know K we can recover the map f from

$$f'(z;\zeta) = \sqrt{\frac{\pi}{K(\zeta,\zeta)}}K(z,\zeta).$$

e) Apply the formula from part d) to the unit disc, and so deduce that

$$f(z;\zeta) = \frac{z-\zeta}{1-\bar{\zeta}z}$$

is the unique function that maps the unit disc onto itself with the point ζ mapping to the origin and with the horizontal direction through ζ remaining horizontal.