Spontaneous Parametric Down Conversion (Yariv "Quantum Electronics")

$\omega_p = \omega_s + \omega_i$: energy conservation

$\overrightarrow{K}_p = \overrightarrow{K}_s + \overrightarrow{K}_i$: Momentum (inside crystal)

Undergraduate E&M: Classical Wave Description

$$\overrightarrow{P} = \chi \overrightarrow{E} \Rightarrow \widetilde{P}(\omega) = \chi(\omega)\overrightarrow{E}(\omega) \Rightarrow P_i(\omega) = \chi_{ij}(\omega)E_j$$

$$P_i^{(\omega)} = \chi_{ij}(\omega)\,\mathrm{E}_j(\omega) + \chi_{ijk}E_jE_k + \ldots\chi_{ijkl}E_jE_kE_l$$

"4-Wave Mixing"

$$\left(e^{i\omega_1 t} + e^{-i\omega_1 t}\right)\left(e^{i\omega_2 t} + e^{-i\omega_2 t}\right) = \left(e^{i(\omega_1+\omega_2)t} + e^{-i(\omega_1+\omega_2)t}\right) + \left(e^{i(\omega_1-\omega_2)t} + e^{-i(\omega_1-\omega_2)t}\right)$$

$P(\omega_1 + \omega_2) \Rightarrow E\left(\omega_1 + \omega_2\right)$: "Sum-frequency generation"

If $\omega_1 = \omega_2$: "Second harmonic generation"

$P(\omega_1 - \omega_2) \Rightarrow E\left(\omega_1 - \omega_2\right)$: "Difference-frequency generation"

Run it backwards: $UV \to IR + IR$

No classical explanation. QM: Vacuum modes stimulate process

$$H = \hbar\omega_p\left(n_p + \frac{1}{2}\right) + \hbar\omega_s\left(n_s + \frac{1}{2}\right) + \hbar\omega_i\left(n_i + \frac{1}{2}\right) + H_i$$

$$H_i :\ X^{(2)}\left(a_p - a^\dagger{}_p\right)\left(a_s - a^\dagger{}_s\right)\left(a_i - a^\dagger{}_i\right)$$

Eight terms $\Rightarrow$ many vanish $\to$ Do not conserve energy

$$a_p a_s^\dagger a_i^\dagger \Rightarrow \omega_p \Rightarrow \omega_s + \omega_i$$

$$a_p^\dagger a_s a_i \Rightarrow \omega_s + \omega_i \Rightarrow \omega_p$$

Treat pump classically: $a_p \to \varepsilon_p e^{-i\omega_p t}$

$$H^i = \hbar G_0\left[a_s^\dagger a_1^\dagger e^{-i\omega_p t} + h.\mathrm{c}.\right]$$

$$G_0 = \sqrt{\omega_s\omega_i}\,\frac{2\pi}{n_s n_i}\,\varepsilon_p\chi^{(2)}\left(\omega_{p,}\omega_{s,}\omega_i\right)\frac{sin\left(\frac{\Delta kl}{2}\right)}{\frac{\Delta kl}{2}}$$

Use $\frac{d\widehat{O}}{dt} = \frac{-i}{\hbar}\left[\widehat{O},H\right] \Rightarrow \left[a_s^\dagger, H\right], \left[a_i^\dagger, H\right]$

$$\Rightarrow a_s^\dagger(t) = \left[a_s^\dagger(0)\cosh G_0 t + i a_i^\dagger(0)\sinh G_0 t\right]e^{i\omega_s t}$$

$$\Rightarrow a_i^\dagger(t) = \left[a_i^\dagger(0)\cosh G_0 t - i a_s^\dagger(0)\sinh G_0 t\right]e^{-i\omega_i t}$$

$$\langle n_s(t)\rangle = \langle n_s(0)\rangle \cosh^2 G_0 t + \left[\langle n_i(0)\rangle + 1\right] sinh^2 G_0 t$$

$$\langle n_i(t)\rangle = \langle n_i(0)\rangle \cosh^2 G_0 t + \left[\langle n_s(0)\rangle + 1\right] sinh^2 G_0 t$$

Response even with no input due to vacuum fluctuations (exactly because $\left[a, a^\dagger\right] = 1$)

For G_0t small, with no input photons, $\left|\Psi^{(t)}\right\rangle = |0\rangle + iG_0 t|1,1\rangle - \frac{(G_0 t)^2}{2}|2,2\rangle + \ldots$

$$P(1,1) = (G_0 t)^2 = \left(\frac{G_o L_n}{c}\right)^2$$

Crystal requirements:

Need non-centro-symmetric crystal

(centrosymmetric crystal has $\chi_{ijk} = 0$ → Need to go for $\chi^{(3)}{}_{ijkl}$)

All systems have $\chi^{(3)}$ but it's even smaller (well, not smaller than 0!)

Usually need birefringent crystal, in order to have phase-matching (momentum conservation) in the presence of crystal dispersion:

$K_p = K_s + K_i$ → $\omega_p n(\omega_p)/c = \omega_s n(\omega_s)/c + \omega_i n(\omega_i)/c$ (using $v = c/n = \omega/k$)

E.g., for degenerate SPDC, $\omega_s = \omega_i = \omega_p/2$,

$\omega_p n(\omega_p) = \omega_p n(\omega_s)/2 + \omega_p n(\omega_i)/2$ → $n(\omega_p) = n(\omega_s)/2 + n(\omega_i)/2$.

This is only true if the index doesn't depend on the frequency/wavelength. To compensate this, we use crystals that have different index of refraction for ordinary and extraordinary polarized light:

E.g., $n_e(\omega_p) = n_o(\omega_s)/2 + n_o(\omega_i)/2$