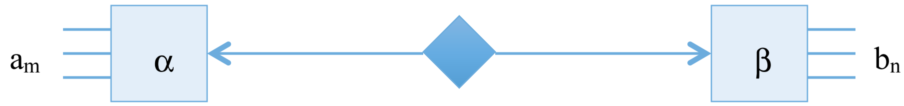
Bell's Inequality (á la Shimony)



$a_m, b_n \in [-1,1]$ "Measurement results"

Define

$p^1(m/\lambda\alpha\beta)$ -- Probability of m'th outcome, given λ, α, β

$p^2(n/\lambda\alpha\beta)$ -- Probability of n'th outcome, given λ, α, β

$p^1(m/\lambda\alpha\beta n)$ -- Probability of m'th outcome, given , λ, α, β ,n

$p^2(n/\lambda\alpha\beta m)$ -- Probability of n'th outcome, given λ, α, β ,m

$p(mn/\lambda\alpha\beta)$ -- Probability of m & n'th outcome, given λ, α, β

Baysian Probability

$$p(mn/\lambda\alpha\beta) = p^1(m/\lambda\alpha\beta n) \bullet p^2(n/\lambda\alpha\beta) \quad \left(= p^2(n/\lambda\alpha\beta m)\, p^1(m/\lambda\alpha\beta)\right)$$

Parameter Independence

$$p^1(m/\lambda\alpha\beta) = p^1(m/\lambda\alpha) \qquad\qquad p^2(n/\lambda\alpha\beta) = p^2(n/\lambda\beta)$$

Outcome Independence

$$p^1(m/\lambda\alpha\beta n) = p^1(m/\lambda\alpha\beta) \qquad\qquad p^2(n/\lambda\alpha\beta m) = p^2(n/\lambda\alpha\beta)$$

Combine $\Rightarrow$ $\boxed{p(mn/\lambda\alpha\beta) = p^1(m/\lambda\alpha) \bullet p^2(n/\lambda\beta)}$ "Bell's locality condition"

Define expectation values:

$$\mathrm{E}^1(\lambda\alpha) = \sum_m p^1(m/\lambda\alpha)\, a_m$$

Deterministic if we set $p^{1,2}(\ \ ) = \{0, +1$

$$\mathrm{E}^2(\lambda\beta) = \sum_n p^2(n/\lambda\beta)\, b_n$$

$$\mathrm{E}(\lambda\alpha\beta) = \sum_{m,n} p(mn/\lambda\alpha\beta)\, a_m b_n = \sum_m p(m/\lambda\alpha)\, a_m \sum_n p(n/\lambda\beta)\, b_n$$

$$= \boxed{\mathrm{E}^1(\lambda\alpha)\mathrm{E}^2(\lambda\beta) = \mathrm{E}(\lambda\alpha\beta)} \text{ -- factorizability}$$

$$\mathrm{E}^1, \mathrm{E}^2 \in [-1, +1]$$

Lemma:

If $\{x,', y, y'\} E[-1,+1]$,

then $-2 \le xy + xy' + x'y - x'y' \le 2$

Proof: 1. S is linear in each of its arguments

$\therefore$ S will be maximum (minimum) at limits $\{x, x', y, y'\} = [\pm1, \pm1, \pm1, \pm1]$

$S_{\substack{\max \\ (\min)}} = 4\,(-4)$

2. $S = (x + x')(y + y') - 2x'y'$

$(0, \pm2)(0, \pm2) - \ (\pm2)$ $\qquad S_{\substack{\max \\ (\min)}} = +2, +6, -2, -6$

$\therefore \quad S_{\substack{\max \\ (\min)}} = +2\,(-2)$

Substitute

$$x = E^1(\lambda\alpha),\ x' = E^1(\lambda\alpha'),\ y = E^2(\lambda\beta),\ y' = E^2(\lambda\beta')$$

and use factorizability

$$xy = E^1(\lambda\alpha)E^2(\lambda\beta) = E(\lambda\alpha\beta)$$

to get $-2 \le E(\lambda\alpha\beta) + (\lambda\alpha\beta') + E(\lambda\alpha'\beta) - E(\lambda\alpha'\beta') \le 2$

We cannot guarantee same λ during an experiment. So we must average over λ:

$E(\alpha, \beta) = \int_\Lambda d\lambda p(\lambda) E(\lambda\alpha\beta) \rightarrow$ Assumption that $\rho(\lambda) \ne \rho(\lambda\alpha\beta)$ "ρ - independence"

where $\int_\Lambda d\lambda \rho(\lambda) = 1$

Then we have our result: $-2 \le S \le 2$ for local, realistic HV theory

$S = E(\alpha\beta) + E(\alpha\beta') + E(\alpha'\beta) - E(\alpha'\beta')$

Experimental requirements

1. Outcome & Parameter independence of results $\Rightarrow$ Use space-like separation, vary $\alpha\ \beta$ rapidly, randomly
2. $\rho - independence: \rho \ne \rho(\alpha\beta) -$ Same method
3. QM must give a violation
4. Measured quantities must provide good approximations to $P_{ens}(mn/\alpha\beta)$-- "detection loophole"