Nonlocality for Two Particles without Inequalities for Almost All Entangled States

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It is shown that it is possible to demonstrate nonlocality for two particles without using inequalities for all entangled states except maximally entangled states such as the singlet state. The eigenvectors corresponding to the measurements that must be performed to do this are exhibited and found to have a particularly simple relationship to the entangled state.

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Bell's 1964 demonstration [1] that realistic interpretations of quantum theory must be nonlocal required the use of inequalities now universally known as Bell inequalities. Greenberger, Horne, and Zeilinger (GHZ) [2] caused much interest when they gave a proof of nonlocality but without using inequalities. Their proof, however, requires a minimum of three particles. A proof of nonlocality without inequalities for two particles had been given earlier by Heywood and Redhead [3] which was much simplified by Brown and Svetlichny [4]. This employed a Kochen-Specker [5] type argument to demonstrate that elements of reality corresponding to space separated measurements must be contextual. The GHZ proof used three spin half particles and the Heywood and Redhead proof used two spin one particles. Thus both proofs required a minimum total of six dimensions in Hilbert space rather than the four required by Bell in his proof. More recently the present author gave a proof of nonlocality for two particles [6] that only requires a total of four dimensions in Hilbert space like Bell's proof but does not require inequalities. This was accomplished by considering a particular experimental setup consisting of two overlapping Mach-Zehnder interferometers, one for positrons and one for electrons, arranged so that if the electron and positron each take a particular path then they will meet and annihilate one another with probability equal to 1. Quantum optical versions of the overlapping interferometers have been proposed in [7] and another version with fermions has been proposed by Yurke and Stoler [8]. The argument has been generalized to two spin s particles by Clifton and Niemann [9] and to N spin half particles by Pagonis and Clifton [10].

So far it has only been shown that this proof can be run for particular entangled states. The purpose of this Letter is to show that it can be run for *any* entangled state except, curiously, maximally entangled states such as the singlet state employed by Bell. We will exhibit the eigenvectors corresponding to the measurements that must be made and find the entangled states which will give the maximum effect. This then is the counterpart of those proofs showing that all entangled states will violate a Bell inequality [11]. (Although in these proofs it is found that the maximally entangled state give the maximum violation of Bell's inequalities.)

By choosing appropriate basis states $|\pm\rangle_i$ for particle i with $i = 1, 2$ (these states do not necessarily have to be associated with spin—they could be associated with any other appropriate physical quantity), any two-particle entangled state can be written in the form (by Schmidt decomposition)

$$|\Psi\rangle = \alpha|+\rangle_1|+\rangle_2 - \beta|-\rangle_1|-\rangle_2, \tag{1}$$

where α and β are two real constants with

$$\alpha^2 + \beta^2 = 1. \tag{2}$$

The minus sign in front of β is chosen for later convenience. Note that we are here considering two particle states for which each particle lives in two dimensions of Hilbert space. For particles living in a higher number of dimensions we could perform a measurement that projects the state of the two particles onto an appropriate four-dimensional subspace and preserves the entanglement and proceed from there. Now we introduce another set of basis states, $|u_i\rangle$ and $|v_i\rangle$ (the notation here is chosen to facilitate comparison with [6]), related to the original basis vectors by

$$|+\rangle_i = b|u_i\rangle + ia^*|v_i\rangle, \tag{3a}$$

$$|-\rangle_i = ia|u_i\rangle + b^*|v_i\rangle, \tag{3b}$$

with inverse relations

$$|u_i\rangle = b^*|+\rangle_i - ia^*|-\rangle_i, \tag{4a}$$

$$|v_i\rangle = -ia|+\rangle_i + b|-\rangle_i, \tag{4b}$$

and

$$|a|^2 + |b|^2 = 1. \tag{5}$$

Note that the orthogonality of the new basis states follows from that of the old basis states. Substituting Eq. (3) into Eq. (1) gives

$$\begin{aligned}|\Psi\rangle = {} & (\alpha b^2 + \beta a^2)|u_1\rangle|u_2\rangle + i(\alpha a^* b - \beta a b^*)|u_1\rangle|v_2\rangle \\ & + i(\alpha a^* b - \beta a b^*)|v_1\rangle|u_2\rangle \\ & - [\alpha(a^*)^2 + \beta(b^*)^2]|v_1\rangle|v_2\rangle.\end{aligned} \tag{6}$$

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For reasons that will become clear later we require that the first term has coefficient equal to zero, i.e., $\alpha b^2 + \beta a^2 = 0$. Thus we can write

$$\frac{a^2}{\alpha} = -\frac{b^2}{\beta} = k^2,$$

or, taking the positive square roots,

$$a = k\sqrt{\alpha}, \quad b = ik\sqrt{\beta}. \tag{7}$$

The solution corresponding to the negative root can be obtained at any stage by putting $\sqrt{\beta} \to -\sqrt{\beta}$. The constant k can be made to be real by choosing the phases of a and b appropriately and we shall assume that this has been done. Thus using Eq. (5) and Eq. (7) we find

$$k^2 = \frac{1}{|\alpha| + |\beta|}. \tag{8}$$

Substituting Eq. (7) into Eq. (6) and using Eq. (8) we obtain

$$|\Psi\rangle = -[\sqrt{\alpha\beta}|u_1\rangle|v_2\rangle + \sqrt{\alpha\beta}|v_1\rangle|u_2\rangle + (|\alpha| - |\beta|)|v_1\rangle|v_2\rangle],$$

which can be written as (dropping the overall factor of -1)

$$\begin{aligned}|\Psi\rangle = &\left(\frac{\sqrt{\alpha\beta}}{\sqrt{|\alpha| - |\beta|}}|u_1\rangle + \sqrt{|\alpha| - |\beta|}|v_1\rangle\right)\\ &\times\left(\frac{\sqrt{\alpha\beta}}{\sqrt{|\alpha| - |\beta|}}|u_2\rangle + \sqrt{|\alpha| - |\beta|}|v_2\rangle\right)\\ &-\frac{\alpha\beta}{|\alpha| - |\beta|}|u_1\rangle|u_2\rangle.\end{aligned} \tag{9}$$

We now introduce a third set of basis vectors defined by

$$|c_i\rangle = A|u_i\rangle + B|v_i\rangle, \tag{10a}$$

$$|d_i\rangle = -B^*|u_i\rangle + A^*|v_i\rangle, \tag{10b}$$

with inverse relations

$$|u_i\rangle = A^*|c_i\rangle - B|d_i\rangle, \tag{11a}$$

$$|v_i\rangle = B^*|c_i\rangle + A|d_i\rangle, \tag{11b}$$

where

$$A = \frac{\sqrt{\alpha\beta}}{\sqrt{1 - |\alpha\beta|}}, \qquad B = \frac{|\alpha| - |\beta|}{\sqrt{1 - |\alpha\beta|}}.$$

Normalization, i.e., that $|A|^2 + |B|^2 = 1$, follows from Eq. (2). Using Eq. (10a) in Eq. (9) we obtain

$$|\Psi\rangle = N(|c_1\rangle|c_2\rangle - A^2|u_1\rangle|u_2\rangle), \tag{12}$$

where

$$N = \frac{1 - |\alpha\beta|}{|\alpha| - |\beta|}.$$

Using Eq. (10a) and Eq. (11a) in Eq. (12) we can write the state of the two particles in the following four equivalent forms:

$$|\Psi\rangle = N(AB|u_1\rangle|v_2\rangle + AB|v_1\rangle|u_2\rangle + B^2|v_1\rangle|v_2\rangle), \tag{13a}$$
$$|\Psi\rangle = N(|c_1\rangle(A|u_2\rangle + B|v_2\rangle) - A^2(A^*|c_1\rangle - B|d_1\rangle)|u_2\rangle), \tag{13b}$$
$$|\Psi\rangle = N((A|u_1\rangle + B|v_1\rangle)|c_2\rangle - A^2|u_1\rangle(A^*|c_2\rangle - B|d_2\rangle)), \tag{13c}$$
$$|\Psi\rangle = N(|c_1\rangle|c_2\rangle - A^2(A^*|c_1\rangle - B|d_1\rangle)(A^*|c_2\rangle - B|d_2\rangle)). \tag{13d}$$

Now consider the physical observables U_i and D_i with corresponding operators,

$$\widehat{U}_i = |u_i\rangle\langle u_i| \text{ and } \widehat{D}_i = |d_i\rangle\langle d_i|,$$

respectively. These physical quantities each can take values 0 and 1 corresponding to the eigenvalues of $\widehat{U}_i$ and $\widehat{D}_i$. Note that $\widehat{U}_i$ and $\widehat{D}_i$ do not, in general, commute so it is not possible, in general, to measure both U_i and D_i on the same particle at the same time. From Eq. (13a) we see that if we measure U_1 and U_2 then

$$U_1 U_2 = 0, \tag{14a}$$

since there is no $|u_1\rangle|u_2\rangle$ term. From Eq. (13b) we see that if we measure D_1 on particle 1 and U_2 on particle 2 then

$$\text{if } D_1 = 1 \text{ then } U_2 = 1, \tag{14b}$$

since only the $|d_1\rangle|u_2\rangle$ term contains $|d_1\rangle$. Similarly, from Eq. (13c) we see that if we measure U_1 on particle 1 and D_2 on particle 2 then

$$\text{if } D_2 = 1 \text{ then } U_1 = 1. \tag{14c}$$

Finally, from Eq. (13d) we see that if we measure D_1 and D_2 then for the experiments

$$D_1 = 1 \text{ and } D_2 = 1 \text{ with probability } |NA^2B^2|^2\ . \tag{14d}$$

The reason that the coefficient of the first term in Eq. (6) was chosen to be equal to zero was in order that we have the prediction (14a).

Using predictions (14a)–(14d) we can prove that realistic interpretations of quantum mechanics are nonlocal. The notion of realism is introduced by assuming that there exist some hidden variables λ which describe the state of each individual pair of particles. Assume that once the state Eq. (1) has been formed the two particles separate and impinge on two distant apparatuses where measurements of U_i or D_i can be made. The assumption of locality is that the choice of measurement on one side cannot influence the outcome of any measurement on the other side. Consider a run of the experiment for which D_1 and D_2 are measured and the results $D_1 = 1$ and $D_2 = 1$ are obtained. That this will happen sometimes follows from (14d). From the fact that we have $D_1 = 1$ it follows from (14b) that if U_2 had been measured we would have obtained the result $U_2 = 1$. If we assume locality then we can assert that, for this particular λ, we would have obtained $U_2 = 1$ even if U_1 had been measured on particle 1 instead of D_1 (because it follows from this assumption that the choice of measurement on particle 1 cannot influence the outcome of any measurement on particle 2). Hence, for this run, U_2 must be determined by the hidden variables to be equal to 1, that is $U_2(\lambda) = 1$. By a similar argument we can deduce from the measurement result $D_2 = 1$ and (14c) that $U_1(\lambda) = 1$. Thus, for this run of the experiment we have $U_1(\lambda)U_2(\lambda) = 1$. Hence, if we had measured U_1 and U_2 instead of D_1 and D_2 then it follows from our assumptions that we would have obtained $U_1U_2 = 1$ but this contradicts (14a). We see that by assuming locality and realism (i.e., hidden variables) we arrive at a contradiction and therefore realistic interpretations of quantum mechanics must be nonlocal. The role played by realism here is that it allows us to assume that there exists some element of reality corresponding to each of U_1 and U_2 even when these quantities are not measured. The elements of reality discussed here can be regarded as Einstein-Podolsky-Rosen elements of reality as their existence is inferred on the basis of predictions of probability equal to 1 [12]. It is interesting to compare this proof with that of GHZ. The GHZ proof can be presented as an "all or nothing" situation: If nature really is local such that quantum mechanics is wrong then in a GHZ type experiment it is necessary that there would be *individual events* that violate the predictions of quantum theory. In Bell's proof it would only be necessary to have a statistical violation of quantum mechanics. The proof presented in this paper falls halfway between these two extremes. If nature is local then either we must have the statistical violation of quantum mechanics that a $D_1 = 1$ and $D_2 = 1$ result is never seen or at least one of the other predictions (14a)–(14c) must be violated in a single event (or both). Thus, once a $D_1 = 1$ and $D_2 = 1$ result is seen, it becomes an all or nothing situation like GHZ. It is also possible to run an argument against Lorentz-invariant realistic interpretations of quantum theory using the predictions (14a)–(14d) and the reader is referred to [6,9,12,13] for details. (Similar though more complicated arguments against Lorentz invariance can be run using a GHZ setup, see [14,15].)

The eigenvectors corresponding to the measurements U_i and D_i can be expressed in terms of the original basis vectors [from Eqs. (4), (7), and (10)]:

$$|u_i\rangle = \frac{1}{\sqrt{|\alpha| + |\beta|}}(\beta^{\frac{1}{2}}|+\rangle_i + \alpha^{\frac{1}{2}}|-\rangle_i), \tag{15}$$

$$|d_i\rangle = \frac{1}{\sqrt{|\alpha|^3 + |\beta|^3}}(\beta^{\frac{3}{2}}|+\rangle_i - \alpha^{\frac{3}{2}}|-\rangle_i), \tag{16}$$

where we have ignored overall factors of magnitude 1. It is interesting that the relationship between the coefficients in Eq. (15) and Eq. (16) and the coefficients in Eq. (1) is particularly simple. If the basis vectors $|\pm\rangle_i$ represent spin $\pm\frac{1}{2}$ along the z direction then

$$|+\rangle_{i\theta} = \cos\frac{\theta}{2}|+\rangle_i + \sin\frac{\theta}{2}|-\rangle_i \tag{17}$$

represents spin $+\frac{1}{2}$ along a direction inclined at an angle θ to the z axis in the x-z plane. Hence, if α and β are positive then we could measure U_i and D_i by measuring spin along directions at angles θ_U and θ_D, respectively, where

$$\tan\frac{\theta_U}{2} = \left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}} \tag{18}$$

and

$$\tan\frac{\theta_D}{2} = -\left(\frac{\alpha}{\beta}\right)^{\frac{3}{2}}, \tag{19}$$

putting $U_i, D_i = 1$ for a spin up result and $U_i, D_i = 0$ for a spin down result.

The nonlocality proof pertains to the fraction $\gamma = |NA^2B^2|^2$ for which $D_1D_2 = 1$. Therefore the maximum nonlocal effect is when this fraction is maximum. Using the above expressions for A, B, and N we can write

$$\gamma = \left(\frac{(|\alpha| - |\beta|)|\alpha\beta|}{1 - |\alpha\beta|}\right)^2. \tag{20}$$

It is easily shown that this has a maximum value of $\frac{1}{2}(5\sqrt{5} - 11)$ (approximately 9%) when $2|\alpha\beta| = 3 - \sqrt{5}$, that is when

$$|\alpha|, |\beta| = 0.9070, 0.4211.$$

For these values we find (taking $|\beta|$ to be the larger number)

$$\theta_U = 68.54^\circ, \quad \theta_D = -35.11^\circ .$$

By considering the negative square root counterpart to Eq. (7), that is, by putting $\sqrt{\beta} \to -\sqrt{\beta}$, we find that we can also use $\theta_U = -68.54^\circ$ and $\theta_D = 35.11^\circ$ as we would expect from symmetry.

If either α or β equals zero then from Eq. (20) we see that $\gamma = 0$ and it will not be possible to run the nonlocality argument. This is to be expected for then the state

Eq. (1) is a product state, that is it is no longer entangled. It is also true that the state in this form would not lead to a violation of the Clauser-Horne-Shimony-Holt (CHSH) inequalities [16]. If $|\alpha| = |\beta|$, then the state Eq. (1) is said to be maximally entangled (the singlet state is one example) and we find that we get the maximum violation of the CHSH inequalities. However, for these values of $|\alpha|$ and $|\beta|$ we find that $\gamma = 0$ and the above nonlocality proof will not go through. The reason for this is that the proof relies on a certain lack of symmetry that is not available in the case of a maximally entangled state.

The experiments that have been proposed with overlapping interferometer would prove quite difficult to realize (although the quantum optical versions in [7] should just be possible). However, it is clear, in the light of these new theoretical results, that an experiment to test this effect would be relatively easy to perform. All that is required is a nonmaximally entangled state [17]. This could be achieved by the methods in [18] and [19] each involving an arrangement of two nonlinear crystals or more easily by modifying an experiment performed by Alley and Shih [20] and also by Ou and Mandel [21]. In this experiment two photons of the same frequency are created by parametric down-conversion and the polarization of one is rotated through 90° and then the two beams are combined at a 50:50 beam splitter. If an unsymmetrical beam splitter was used instead then a nonmaximally entangled state would be produced. Note that in the case of polarizations the above angles would have to be halved.

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