**PHYS514** **Homework 3**
**Fall 2013** **due 10/3**

## 1. Magnetic Resonance

Consider a (spin-1/2) electron in a static magnetic field $\vec{B}_0 = B_0\,\hat{z}$. The state of the spin at $t = 0$ is $|\psi_0\rangle = (|+\rangle - |-\rangle)\,/\,\sqrt{2}$.

a. What is the fictitious classical spin $\vec{S}$ associated with $|\psi_0\rangle$?

b. At $t = 0$, an oscillating magnetic field with amplitude $B_\perp = \hbar\omega_l/200\mu$ is applied ($\omega_l$ is the Larmor frequency and $\mu$ is the magnetic moment). Solve the classical equations of motion for $\vec{S}$ (in the RWA) if…

   i. $\vec{B}(t) = B_\perp \cos(\omega_l t)\,\hat{x}$ What other choice for $|\psi\rangle$ would give rise to the same kind of time dependence for $\vec{S}$?

   ii. $\vec{B}(t) = B_\perp \cos(\omega_l t)\,\hat{y}$

   iii. $\vec{B}(t) = B_\perp\,[-\sin(\omega_l\,t)\,\hat{x} + \cos(\omega_l t)\,\hat{y}\,]$ How and why is this answer different than what you found in part ii?

c. For each case in part b, make an accurate plot / sketch of the probability of finding the electron in the $|-\rangle$ state vs. time.

## 2. Beyond the RWA

Consider a (spin-1/2) electron in a magnetic field $\vec{B} = B_1\cos(\omega_0 t)\,\hat{x} + B_0\hat{z}$, where $\omega_0 = -2\,\mu B_0/\hbar$.

a. Writing the state of the electron as $|\psi\rangle = a_+|+\rangle + a_-|-\rangle$, what are the equations of motion for $a_+$ and $a_-$ (from the time-dependent Schrodinger equation)?

b. Make the RWA, and solve the equations from a given the initial condition $a_+(t=0) = 1, a_-(t=0) = 0$.

c. Now, take the equations from part a and solve them without making the RWA, given the initial condition $a_+(t=0) = 1, a_-(t=0) = 0$. Either solve the equations analytically using special functions, or solve them numerically (e.g., using NDSolve in Mathematica). Make an accurate plot of the solution for $B_1/B_0 = 0.01$ and $B_1/B_0 = 0.1$. What is a physical explanation (using classical magnetic resonance) for the difference between the answer for part c and b?

## 3. Matrix Elements

A $^{133}$Cs atom is in the $|F = 3, m_F = 0\rangle$ state in a magnetic field $\vec{B} = 1\,Gauss\,\hat{z}$. Assume that the Zeeman effect is linear (i.e., in the regime $\mu B << \Delta E_{hfs}$). A magnetic field $\vec{B} = 0.01\,Gauss\cos\left(\frac{\Delta E_{hfs}}{\hbar}t\right)(\hat{x} + \hat{z})$ is applied, where $\Delta E_{hfs}$ is the ground-state-hyperfine splitting for zero magnetic field. Which $m_f$ states in the $F = 4$ hyperfine state can the atom make a transition to? What are the effective Rabi rates $\Omega_{eff} = \sqrt{4\Omega^2 + \delta^2}$ for those transitions?

In the next two problems, pick a convention for the gyromagnetic ratio (i.e., positive or negative). Also, work only in the rotating frame.

## 4. Quantum projection noise

A spin-1/2 particle initialized in the $|+\rangle$ state is used in a Ramsey experiment as a clock. A magnetic field $\vec{B} = B_0\hat{z}$ is always present, and a field $\vec{B} = B_\perp \cos(\omega t)\,\hat{x}$ is used for the $\pi/2$ pulses. Assume that the detuning is small and that the $\pi/2$ pulses are instantaneous, so that the fictitious spin lies in the x-y plane after the first $\pi/2$ pulse.

a. What is the spin state after the free evolution time $T$ and before the second $\pi/2$ pulse as a function of the detuning $\delta = \omega - \omega_0$, where $\omega_0 = -2\,\mu B_0/\hbar$?

b. One way to think about the purpose of a clock experiment is to find the center of the peak that appears at $\delta = 0$. A method for accomplishing that is to determine the frequencies at which the probability $P_-$ to find the particle in $|-\rangle$ first falls to ½ as $\omega$ is tuned just higher and lower than $\omega_0$. In the simple Ramsey experiment considered here, those frequencies are $\omega_u = \omega_0 + \pi/\,2\,T$ and $\omega_l = \omega_0 - \pi/\,2\,T$.
   i. What is the quantum state after the second $\pi/2$ pulse for these two frequencies?
   ii. How is $P_-$ related to $\langle S_z\rangle$?
   iii. What is the variance $\Delta_-^2$, therefore, in $P_-$ after the second $\pi/2$ pulse for $\omega = \omega_0 + \pi/\,2\,T$ and $\omega = \omega_0 - \pi/\,2\,T$?

c. Now, let's average over $N$ atoms. The uncertainty in $P_-$ averaged over $N$ measurements is the standard error of the mean: $\Delta_-/\sqrt{N}$. Imagine that we do the following experiment. We know that $P_- = \frac{1}{2} + \frac{1}{2}\cos(\delta\,T)$ for small detunings. We find $\omega_u$ and $\omega_l$ by measuring $P_-$. Then we find $\omega_0 = (\omega_u + \omega_l)/2$. Now, we want to know our uncertainty in $\omega_0$. Using error propagation, what is our uncertainty in $\omega_0$? Even in a perfect, noiseless experiment, this quantum projection noise limits the precision with which we can find the clock frequency. Note that certain entangled states can improve on this limit.

## 5. Spin Echo

A spin-1/2 particle initialized in the $|+\rangle$ state is used in a Ramsey experiment. We average each measurement of $P_-$ over $N$ runs of the experiment. In each run of total evolution time $T$, a magnetic field $\vec{B} = (B_0 + \delta B_0)\,\hat{z}$ is always present, and a field $\vec{B} = B_\perp \cos(\omega_0 t)\,\hat{x}$ is used for the $\pi/2$ pulses (where $\omega_0 = -2\,\mu B_0/\hbar$). Assume that the $\pi/2$ pulses are instantaneous and that $\delta B_0$ is very small, so that the fictitious spin lies in the x-y plane after the first $\pi/2$ pulse.

a. If the distribution of $\delta B_0$ is uniform between $-B_0/100$ and $B_0/100$, then what is the distribution of the fictitious spin associated with the spin-1/2 particle before the second $\pi/2$ pulse?

b. Given this distribution, what is the distribution of $\langle S_z\rangle$ and $P_-$ after the second $\pi/2$ pulse? What is $P_-$ averaged over all runs?

c. The same measurement is carried out, but with an instantaneous $\pi$ pulse inserted at $T/2$. In this case, what is the distribution of fictitious spin associated with the spin-1/2 particle before the second $\pi/2$ pulse? What is the distribution of $\langle S_z\rangle$ and $P_-$ after the second $\pi/2$ pulse? What is $P_-$ averaged over all runs?