PHYS 515: Homework Set 8

March 31, 2020

Due Date: Tuesday April 7, 2020, at the beginning of class.

Topic: Graduate General Relativity 1

1. Derive the field equations in vacuum from the Einstein-Hilbert action shown in class by showing the following steps. All steps must be shown, even if there is overlap with what was done in class.

   (a) Prove the variation of the square root of the determinant of the metric is

   $$\delta g = g\left(g^{\mu\nu}\delta g_{\mu\nu}\right) .$$

   This was done in class, but all the steps should reproduced for this homework.

   (b) Prove the variation of the Riemann tensor is

   $$\delta R^{\rho}{}_{\mu\lambda\nu} = \nabla_{\lambda}\left(\delta\Gamma^{\rho}_{\nu\mu}\right) - \nabla_{\nu}\left(\delta\Gamma^{\rho}_{\lambda\mu}\right) .$$

   (c) Prove the variation of the Christoffel symbol is

   $$\delta\Gamma^{\sigma}_{\mu\nu} = -\frac{1}{2}\left[2g_{\lambda(\mu}\nabla_{\nu)}\left(\delta g^{\lambda\sigma}\right) - g_{\mu\alpha}g_{\nu\beta}\left(\nabla^{\sigma}\delta g^{\alpha\beta}\right)\right] .$$

   (d) With parts (b) and (c), prove that the total variation of the Riemann tensor vanishes

   $$\delta S = \int \sqrt{-g}d^4x g^{\mu\nu}\delta R_{\mu\nu} = \int \sqrt{-g}d^4x g^{\mu\nu}\delta R^{\rho}{}_{\mu\rho\nu} = 0 .$$

   (e) Finally, put all of these steps together, along with what was done in class, to derive Einstein's equation.

2. Carroll 4.2

   We showed how to derive Einstein's equation by varying the Hilbert action with respect to the metric. They can also be derived by treating the metric and connection as independent degrees of freedom and varying separately with respect to them; this is known as the **Palatini formulation**. That is, consider the action

   $$S = \int d^4x\sqrt{-g}g^{\mu\nu}R_{\mu\nu}(\Gamma) ,$$

   where the Ricci tensor is thought of as constructed purely from the connection, not using the metric. Variation with respect to the metric gives the usual Einstein's equations, but for a Ricci tensor constructed from a connection that has no a priori relationship to the metric. Imagining from the start that the connection is symmetric (torsion free), show that variation of this action with respect to the connection coefficients leads to the requirement that the connection be metric compatible, that is, the Christoffel connection. Remember that Stokes's theorem, relating the integral of the covariant divergence of a vector to an integral of the vector over the boundary, does not work for a general covariant derivative. The best strategy is to write the connection coefficients as a sum of the Christoffel symbols $\tilde{\Gamma}^{\lambda}_{\mu\nu}$ and a tensor $C^{\lambda}{}_{\mu\nu}$,

   $$\Gamma^{\lambda}_{\mu\nu} = \tilde{\Gamma}^{\lambda}_{\mu\nu} + C^{\lambda}{}_{\mu\nu} ,$$

   and show that $C^{\lambda}{}_{\mu\nu}$ must vanish.

3. Carroll 4.5

   A spacetime is static if there is a timelike Killing vector that is orthogonal to space- like hypersurfaces. Generally speaking, if a vector field v^μ is orthogonal to a set of hypersurfaces defined by $f =$ constant, then we can write the vector as $v_\mu = h\nabla_\mu f$ (here both f and h are functions). Show that this implies

$$v_{[\sigma}\nabla_\mu v_{\nu]} = 0 \ .$$

4. Express the Maxwell Equations in terms of the 4-potential A^μ and show that the straightforward application of the minimal coupling prescription (look it up) leads to two different possible generalizations of the Maxwell Equations to curved space. Show that one of these two generalizations leads to a violation of charge conservation. In an Earth-bound laboratory what would the typical magnitude of this violation of charge conservation be? Are charge conservation violations of this magnitude ruled out by experiment or observations? Discuss.