Physics 524 Week 2 Homework

Due: Tuesday 9/5/2023 at 10am

Due date reminder, etc.

Please email your completed assignment to the course TA by Tuesday, 10 am of next week. Assignments that are late by at most one week will receive at most 50% of full credit. We will not grade anything submitted more than one week late.

Your homework submissions—code, cell phone photos, etc. must include enough identifying information for us to tell who you are!

Problem 1: A much better infinite series for π

In class we worked on an arctan series to evaluate π. A much more rapidly converging series was discovered by the brilliant Indian mathematician Srinivasa Ramanujan[1]. It is

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{k=0}^{\infty}\frac{(4k)!\ (1103 + 26390k)}{(k!)^4\ 396^{4k}}\ ,$$

where *k!* ("*k* factorial") is 1×2 ×3 ×... ×*k* and (4*k*)! = 1×2 ×3 ×... ×4*k*.

Please write a Python script that calculates an approximation to π using the Ramanujan series, and comment on its accuracy after 1, 2, and 3 terms. (Recall that the value of π is 3.14159265358979323846264338327950288419716939937510582…)

Note that there are even faster-converging formulas than this! One, mentioned in Wolfram MathWorld,[2] adds 50 digits of precision for each additional term.

For full credit, include your code and print out your three approximations and how each compares to π.

Problem 2: Relativistic spaceflight

As undergraduates you probably learned about some of the surprising consequences of special relativity: a moving clock ticks slowly, a moving object becomes shorter along its direction of motion, the velocity addition formula never yields a superluminal[3] speed.

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[1] https://en.wikipedia.org/wiki/Srinivasa_Ramanujan
[2] http://mathworld.wolfram.com/PiFormulas.html
[3] cool word, isn't it?

Imagine that we have two frames of reference, which I'll call *O* and *O'*. ("*O*" stands for "origin," I suppose.) Frame *O* is at rest with respect to the earth, while *O'* is fixed to a starship coasting at constant speed $v_{starship}$ along the positive x axis, according to observers on earth.

Though identical clocks on earth and on the starship were manufactured to tick once per second, earth observers will see the starship's clocks ticking slowly. When an earth clock measures a time interval Δt, a starship clock will measure a shorter interval Δt' with

$$\Delta t' = \Delta t\sqrt{1 - \frac{v_{starship}^2}{c^2}}\ .$$

Imagine that the starship launches a shuttle craft in the positive *x* direction, moving with speed *u* with respect to the starship. Naturally, earth observers will see the shuttle moving faster than *u* since the starship is already moving in the *x* direction. We use the relativistic velocity addition formula to calculate the shuttle's speed as seen by observers on earth:

$$v_{shuttle} = \frac{u + v_{starship}}{1 + \frac{u v_{starship}}{c^2}}\ .$$

It is easy to show that in the limit that $v_{starship}$ and *u* approach *c*, $v_{shuttle}$ also approaches (but does not exceed) *c*. Note that the ***exact*** value of the speed of light is *c* = 299,792,458 m/s.

Let's use the time dilation and velocity addition formulas to analyze the motion of a starship undergoing the uniform *proper* acceleration $g = 9.81\ m/s^2$. By "proper acceleration" I mean the acceleration sensed by someone on the starship. Naturally, an observer on earth will see that the starship never exceeds the speed of light, so its acceleration (according to earth observes) will approach zero.

We *could* work up an analytic description of $x_{starship}(t)$ and $v_{starship}(t)$ by doing some integrals[4]. But why not just break the flight time up into small time intervals and sum the changes in position and velocity in a loop? That's the kind of thing computers are good at.

We'd like to answer the following questions, assuming the ship starts at rest near the earth. After a certain amount of time in space, how far has the ship gone, according to earth observers? How fast is it moving? How much time has passed on the earth and starship clocks?

The key is to keep in mind that during ten second of ship's time (corresponding to a longer time interval, according to an earth observer), the ship's speed has increased by 98.1 m/s according to a cosmonaut on the starship. As a result, earth observers will see that the ship's velocity has increased from $v_{starship}$ to

$$\frac{98.1 + v_{starship}}{1 + \frac{98.1 v_{starship}}{c^2}}\ .$$

During ten seconds of ship's time, earth observers will see that the ship has moved a distance that is approximately equal to the product of the ship's speed at the ***start*** of the interval and the ***duration*** of the interval, *according to earth clocks*:

[4] You can look up the analytic solution on Wikipedia: https://en.wikipedia.org/wiki/Hyperbolic_motion_(relativity)

$$\Delta x \approx v_{starship} \frac{10 \text{ seconds}}{\sqrt{1 - \frac{v_{starship}^2}{c^2}}}.$$

(I am using the approximation that the time dilation factor is constant during the short time interval.)

Please do the following. Consider a one-way voyage of four year's duration as measured by a clock on the starship. (Take the length of a year to be exactly 365.25 days, where one day is 24 hours long.) By breaking the outbound voyage into ten second intervals (***as measured by the starship clock***), write a program that calculates how far the starship has traveled (according to earth observers), how fast it is moving (according to earth observers), and how much a clock on the earth has advanced, at the end of the voyage. You will do this by looping over elapsed time intervals, each of ten second's ship time duration.

Your code should update the position of the ship, then the reading on the earth clock, then the velocity as seen by earth during each pass through the loop. To keep track of what your program is doing, have it print out regular updates of the ship's position and velocity (as measured in the earth frame), as well as the elapsed time in both frames.

Here's what I mean. If at some particular time the ship is traveling at 0.6 *c* according to earth observers, then 10 seconds of ship time will correspond to 12.5 seconds of earth time. In 12.5 seconds of earth time the ship will move approximately 12.5 × 0.6 × 299,792,458 meters. At the end of the interval the ship's new velocity will be

$$\frac{98.1 + 0.6c}{1 + \frac{(98.1)(0.6c)}{c^2}}.$$

Keep in mind that the ship's calendar includes a leap year, so that there are 1461 days, or 126,230,400 seconds of ship's time in the four-year voyage.

For your information: my version of the program yields these answers:

final ship time (weeks) 208.71428571428572

final earth time (weeks) 1571.41469651

final ship time (seconds) 126230400.0

final earth time (seconds) 950391608.447

final ship speed (% c) 99.94834273779534

final ship distance (lightyears) 29.1632735122

If your code is correct it will agree with mine to splendidly impressive precision. If your code isn't correct, then dig into it with the iPython debugger to look for problems.

For full credit, determine (and print) the final earth time along with final ship speed and time for a four year journey as shown above.