Vibrational Analysis

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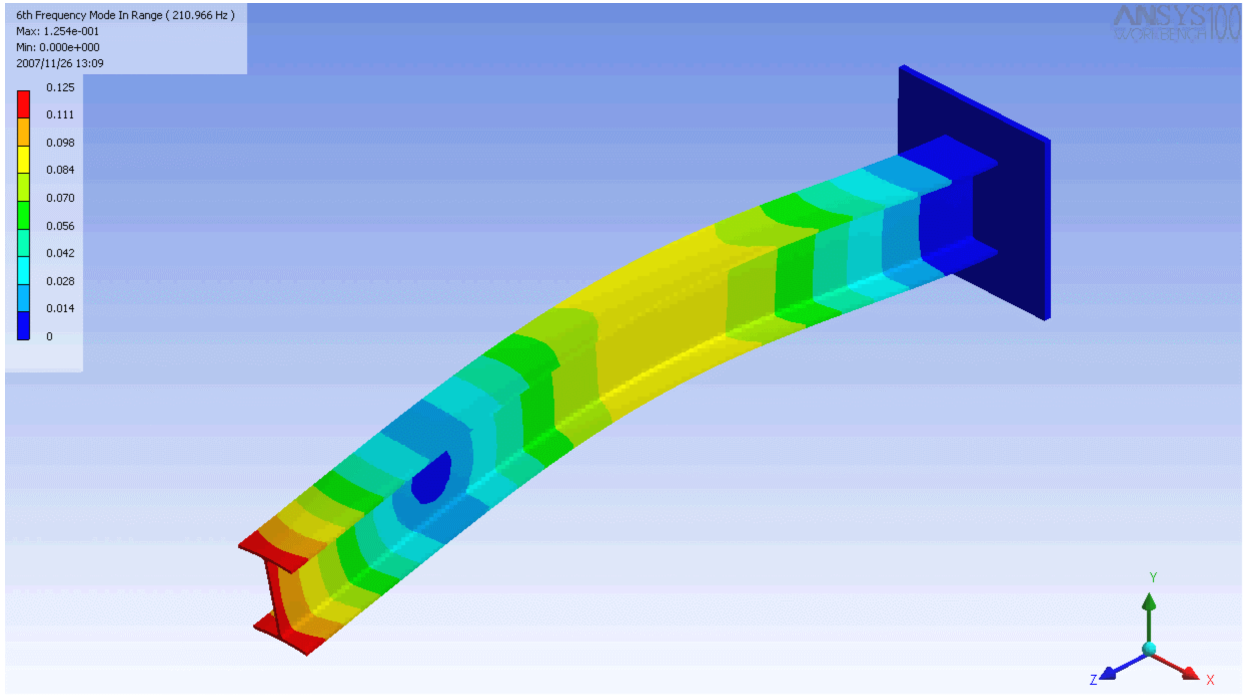
Modal analysis

Modal (frequency) analysis: FEA can be used to predict the deformation modes associated with vibrations (natural frequencies)

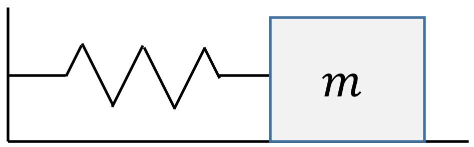


Knowing the natural vibration frequencies can be helpful, for example, to avoid loading a structure at its natural resonance frequencies

In theory, a structure has an infinite number of natural frequencies. Generally, the lowest several are most readily observed and are of the greatest interest

Natural frequency of an undamped spring

Consider an undamped spring-mass system with no external loads or friction. Force equilibrium on the system gives:



$$\sum F_x = m\ddot{u} = -ku$$

$$\rightarrow m\ddot{u} + ku = 0$$

This equation can be re-cast into the form below by dividing both sides of the equation by the mass m:

$$\ddot{u} + \omega_n^2 u = 0, \qquad \text{where} \quad \omega_n = \sqrt{\frac{k}{m}}$$

This second-order ODE has a general solution $u = A\cos(\omega_n t - \gamma)$, where the quantity ω_n is called the **natural angular frequency**.

The angular frequency ω_n has units of radians per unit time, e.g. rad/sec. The frequency f measures cycles per unit time e.g. Hertz (Hz), or cycles per second. The relationship between these quantities if $\omega = 2\pi f$.

The amplitude A is **arbitrary**. This analysis only tells us what frequency the system is predisposed to vibrate at, **not** the amplitude at which the vibrations will occur (the latter requires specifying specific boundary conditions).

Free undamped vibrations in FEA

In the previous example, we had a single degree of freedom: the displacement of the mass on the end of the spring.

In an FEA model, we have many degrees of freedom (e.g. nodal displacements). It can be shown that an FEA model with free undamped vibrations results in a **system of equations** of the form

$$\boldsymbol{M}\{\ddot{\boldsymbol{U}}\} + \boldsymbol{K}\{\boldsymbol{U}\} = \boldsymbol{0}$$

$\boldsymbol{K}$ is the **stiffness matrix** of the system (same as in static analysis)

$\boldsymbol{M}$ is called the **mass matrix**, and depends on the elements being used and the mass density of the material in the model.

$\{\boldsymbol{U}\}$ contains the nodal degrees of freedom.

To find natural vibration modes and frequencies of the system, we assume a displacement solution of $\{\boldsymbol{U}\} = \{\boldsymbol{\Phi}\}e^{i\omega t}$

With this assumption, we can write $\{\ddot{\boldsymbol{U}}\} = -\omega^2\{\boldsymbol{\Phi}\}e^{i\omega t}$. Substituting into above, we obtain

$$[\boldsymbol{K} - \omega^2\boldsymbol{M}]\{\boldsymbol{\Phi}\} = \boldsymbol{0}$$

This only has an interesting solution if $\det(\boldsymbol{K} - \omega^2\boldsymbol{M}) = 0$ (eigenvalue problem)

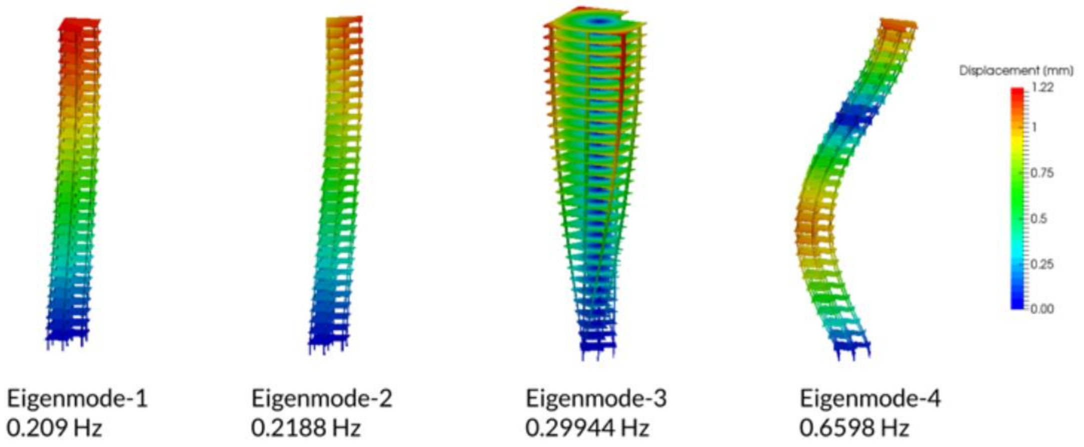
For matrices of dimension $n \times n$ there will be n values of ω_i^2 that satisfy the eigenvalue equation above, and hence n vectors $\{\boldsymbol{\Phi}_i\}$. These represent the natural vibration frequencies and deformation modes, respectively. Here, $n =$ number of unconstrained DOFs in the system

Free undamped vibrations in FEA

Natural vibration modes are possible frequencies and deformation that a structure can take on. Modal analysis gives the deformation shapes associated with each frequency. **There are as many frequencies as DOFs in the model**.

Different modes may be more active and prominent for different kinds of boundary conditions. Typically, the first ~6 lowest frequency modes will be most prominently observed (Thomas J. R. Hughes, *The Finite Element Method: Linear Static and Dynamic Finite Element Analysis*)

The magnitudes of displacement from such findings are **arbitrary**; you'd have to apply specific force boundary conditions to know how **severely** a structure will actually deform



https://www.simscale.com/docs/simwiki/fea-finite-element-analysis/what-is-natural-frequency/

Modal analysis: Unconstrained structure

Unlike static structural analysis, it is possible to perform modal analysis with no displacement or rotation constraints on any node.

In this case, you will obtain several zero-frequency values, which are associated with rigid body modes (rigid translation or rotation through space, no stress/deformation in the body)

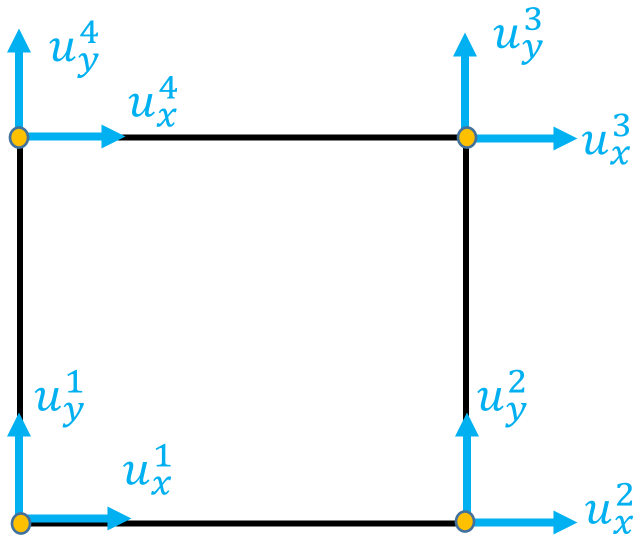
Dimensionality	Rigid translation modes	Rigid rotation modes	Total rigid body modes
1	1	0	1
2	2	1	3
3	3	3	6

Modal analysis example

Recall the 4-node quadrilateral (Q4) element that we discussed in a previous lesson.

Suppose we model a structure using a **single Q4 element**. How many vibrational frequencies would be obtained from each of the boundary conditions listed below, and, how many of those frequencies would be associated with a rigid-body mode (and hence zero-valued):

1. All DOFS on the left edge of the element are fixed (cantilever-style support)
2. Node 3 displacements are fixed, all others are unconstrained
3. No displacement boundary conditions are applied to the structure



Dimensionality	Rigid translation modes	Rigid rotation modes	Total rigid body modes
1	1	0	1
2	2	1	3
3	3	3	6

Modal analysis example

You wish to use FEA to find the lowest natural frequency on the tuning fork pictured below. You create a fully 3D model without any physical constraints/supports on the fork in order to find the lowest frequency associated with free, unconstrained vibration.

Dimensionality	Rigid translation modes	Rigid rotation modes	Total rigid body modes
1	1	0	1
2	2	1	3
3	3	3	6

The FEA software you use asks you how many vibrational frequencies you would like it to compute for your model. What is the minimum number you should enter?

Modal Analysis Summary

To find the natural vibration frequencies of an FEA model:

1. Create mesh, specify material properties, and impose any displacement boundary conditions that constrain the structure of interest. **Mass density is a required material property for this analysis** (this parameter is not needed for static structural analysis)

2. Perform a frequency (eigenvalue) analysis using FEA software of your choice. Be sure to request a reasonable number of frequencies to be reported. Generally, the lowest ~6-10 modes (not associated with rigid body motion) are of interest