An introduction to fluid dynamics

I'm going to follow the development of the subject as presented in *Mechanics*, 3rd edition, K. R. Symon, Addison-Wesley Publishing, 1971. See chapter 8, sections 6-9.

Another good reference is *Lectures in Elementary Fluid Dynamics: Physics, Mathematics and Applications*, J.M. McDonough, Departments of Mechanical Engineering and Mathematics, University of Kentucky, Lexington, KY (2009): http://www.engr.uky.edu/~acfd/me330-lctrs.pdf .

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Utility of conservation laws in fluid dynamics

Many (most ??) of the useful equations in fluid dynamics come about because of various *conservation laws*. We'll only deal with non-relativistic fluids, so we'll always have one set of equations which comes about because mass is conserved; if the density in a fluid at some point in space increases/decreases, it must be associated with a net inflow/outflow of stuff from that point in space.

Schematically:

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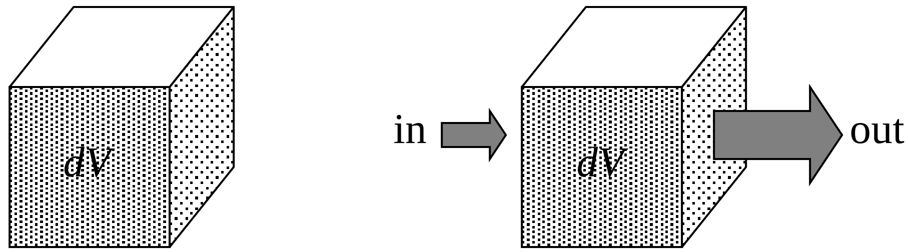

$dm = \rho dV$

dm decreases

Other conservation laws which might be useful:

- Perhaps momentum conservation? We'd want to work out a way to say something like this: "momentum inside the volume element dV can only change when (more/less) momentum enters the box than leaves it."

- Maybe conservation of energy? This one's more complicated since if you compress some collection of particles in the fluid, you do work on them, increasing their potential energy. Also, if the fluid flows uphill (against the earth's gravitational field), its potential energy changes.

An added piece of complication comes about because there are two kinds of derivatives that we'll be interested in. We might want to know how, for example, the pressure at a *fixed point* changes with time. But we also might want to know about the rate of change of the pressure at a point which moves along with the fluid. For example, a "chunk" of air moving over an airfoil will show condensation fog if it is humid and the pressure drops suddenly, cooling the air.

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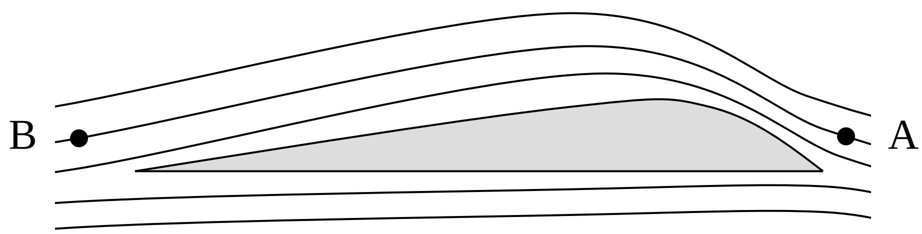
Partial derivatives and convective derivatives

This is a natural place to consider the difference between partial and total derivatives.

"Laminar" (non-turbulent, layered) air flow over a wing might look like this:



At the point A, fixed to remain in front of the airfoil, the pressure will be constant in time. The same is true at point B. Since we're talking about holding x, y, z constant, if the pressure is a function of x, y, z, t (let's call it $P(x, y, z, t)$), we can write

$$\left.\frac{\partial P}{\partial t}\right|_{\text{point A}} = 0 \quad \text{and} \quad \left.\frac{\partial P}{\partial t}\right|_{\text{point B}} = 0 .$$

Since *P* is a function of four variables, we need to specify (by taking a partial derivative) that all but *t* are being held fixed in our description of the behavior of the pressure at points A and B.

A group of air molecules in a small volume element that flows from point A to point B will experience a changing pressure since the local air pressure goes up as the air "gets crowded" in front of the air foil's leading edge then drops as the air moves over the upper surface of the wing. We would like to be able to describe this kind of thing too. Let's investigate.

From the chain rule, the total derivative of *P* is

$$\frac{dP}{dt} = \frac{\partial P}{\partial t} + \frac{\partial P}{\partial x}\frac{dx}{dt} + \frac{\partial P}{\partial y}\frac{dy}{dt} + \frac{\partial P}{\partial z}\frac{dz}{dt} .$$

Note the partials of *P*, but the total derivatives for *x*, *y*, *z* with respect to time. What we're doing, in effect, is declaring how we want to move around in space by saying we'll cruise around with the same velocity as the average fluid velocity of a particular tiny fluid packet. We'll then figure out how P changes with time in the packet which moves with (changing) velocity

$$\vec{v} = \frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z}.$$

Recall: $\frac{\partial P}{\partial x}\hat{x} + \frac{\partial P}{\partial y}\hat{y} + \frac{\partial P}{\partial z}\hat{z} = \vec{\nabla} P$ (see the math review)

As a result, we can write

$$\frac{dP}{dt} = \frac{\partial P}{\partial t} + \vec{v} \cdot \vec{\nabla} P.$$

This is handy: it lets us say how the pressure changes if we move around with velocity $\vec{v}$ inside the fluid volume. Sometimes (usually??) we'll choose the velocity to be the same as the local fluid flow velocity. Symbolically, we can write

$$\boxed{\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}.}$$

This quantity is nothing more than the total (time) derivative operator for a function that depends on x, y, z, and t. It is sometimes referred to as the "convective derivative," "Stokes derivative," "Lagrangian derivative," or any one of a number of other names.

More on this: imagine we're in a submarine, measuring the water temperature $T(x,y,z,t)$ as we motor along. If we come to a stop and measure how the temperature changes with time, we'll be determining the partial derivative of T: $\partial T/\partial t$ since we're holding x, y, z fixed by halting the submarine.

If we don't stop our submarine, the rate of change of the temperature will depend on how much the temperature varies from place to place (and how quickly we're moving around), as well as any built-in time dependence, for example, from the sun heating the ocean during the day.

Taking both the "built-in" and position-related effects into account, we'll measure

$$\frac{dT(x,y,z,t)}{dt} = \frac{\partial T(x,y,z,t)}{\partial t} + (\vec{v}\cdot\vec{\nabla})T(x,y,z,t)$$

with $\vec{v}$ the velocity of our submarine.

We might want our submarine to cruise along with the same velocity as the water around us (in that case, we'll get to observe the behavior of the temperature of the same water molecules all day long), but we aren't required to do this for the above equation to hold.

If the sun isn't shining (so there's nothing heating the water) and the ocean current flows steadily, without change (so we don't suddenly have cold water from the bottom blasting past us), we'll expect the temperature at a fixed point in the ocean to remain unchanged. In that case, $\partial T/\partial t = 0$ so that we'll only sense a change in the temperature if we change our position:

$$\frac{dT(x,y,z,t)}{dt} = \vec{v} \cdot \vec{\nabla} T(x,y,z,t).$$

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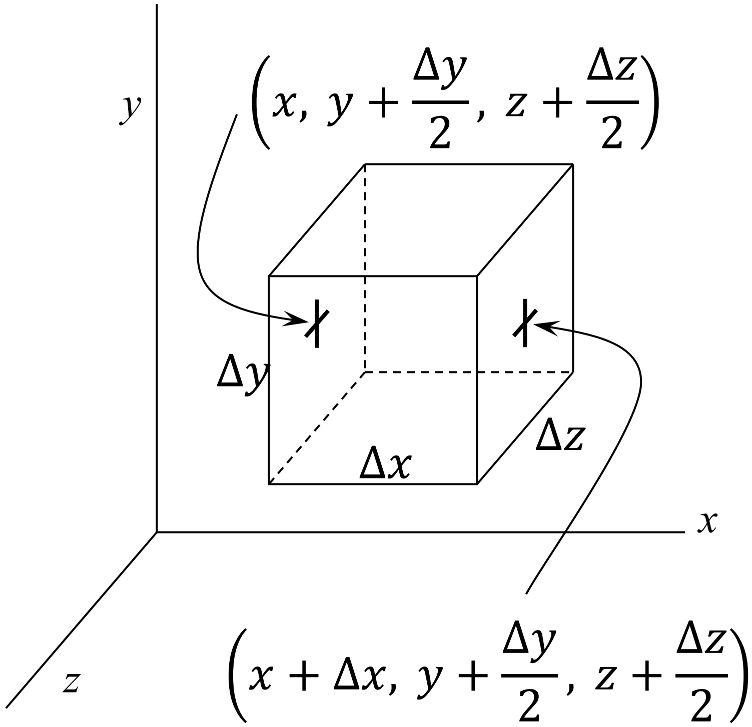
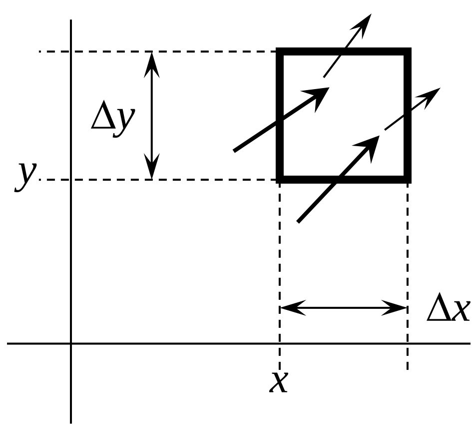
Conservation of mass

Let's work up a differential equation that expresses the idea that mass is neither created nor destroyed in our fluid.

We will investigate the inflow/outflow of mass in a small volume $dV = dxdydz$ at the point *x*, *y*, *z*. (I'm only going to draw it in two dimensions, to simplify the picture.)



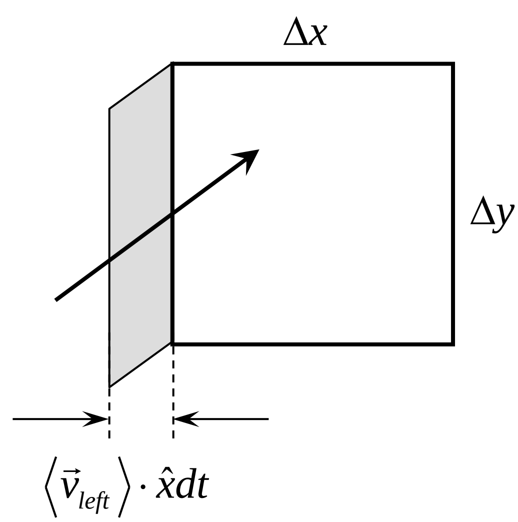


I've drawn the fluid flow so that the fluid's velocity *into* the box from the bottom and left sides is greater than the velocity *out* of the box through the top and right sides. As a result, we expect fluid to build up in the box, so the density should increase.

We can take as the average fluid velocity for the fluid that enters the left side of the box the exact fluid velocity at the center of the left side:

$$\left\langle \vec{v}_{left} \right\rangle \approx \vec{v}\left(x, y + \frac{\Delta y}{2}, z + \frac{\Delta z}{2} \right).$$

The volume of fluid that flows into the box from the left side during time *dt* is shown as a shaded region in the following diagram.



The volume of the region of fluid that flows into the left side of the box is approximately

$$\left[\left\langle \vec{v}_{left} \right\rangle \cdot \hat{x} dt\right] \Delta y \Delta z \approx \vec{v}\left(x, y + \frac{\Delta y}{2}, z + \frac{\Delta z}{2} \right) \cdot \hat{x} dt \Delta y \Delta z$$

since the thickness of it in the x direction is $\vec{v}\ dt \cdot \hat{x}$.

The volume that flows in from the bottom is

$$\vec{v}\left(x + \frac{\Delta x}{2}, y, z + \frac{\Delta z}{2} \right) \cdot \hat{y} dt \Delta x \Delta z\,.$$

The volume that flows out the top is

$$\vec{v}\left(x + \frac{\Delta x}{2}, y + \Delta y, z + \frac{\Delta z}{2} \right) \cdot \hat{y} dt \Delta x \Delta z.$$

The volume that exits the right side is

$$\vec{v}\left(x + \Delta x, y + \frac{\Delta y}{2}, z + \frac{\Delta z}{2} \right) \cdot \hat{x} dt \Delta y \Delta z\,.$$

The *mass* which flows in from the left is the product of the local density and the in-flowing volume:

$$dm_{left} \approx \rho\left(x, y + \frac{\Delta y}{2}, z + \frac{\Delta z}{2}, t \right) \vec{v}\left(x, y + \frac{\Delta y}{2}, z + \frac{\Delta z}{2}, t \right) \cdot \hat{x} dt \Delta y \Delta z$$

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Note that I'm approximating density and velocity for the left side using the value of ρ and $\vec{v}$ at the center of the left face of the volume element.

Adding up the in-flowing and out-flowing mass for all six faces of the volume element (and omitting explicit time dependence, to save myself the effort of writing ", t" everywhere) gives

$$\begin{aligned}
dm = {} & \rho\left(x, y+\frac{\Delta y}{2}, z+\frac{\Delta z}{2}\right) v_x\left(x, y+\frac{\Delta y}{2}, z+\frac{\Delta z}{2}\right) dt\Delta y\Delta z \\
& -\rho\left(x+\Delta x, y+\frac{\Delta y}{2}, z+\frac{\Delta z}{2}\right) v_x\left(x+\Delta x, y+\frac{\Delta y}{2}, z+\frac{\Delta z}{2}\right) dt\Delta y\Delta z \\
& +\rho\left(x+\frac{\Delta x}{2}, y, z+\frac{\Delta z}{2}\right) v_y\left(x+\frac{\Delta x}{2}, y, z+\frac{\Delta z}{2}\right) dt\Delta x\Delta z \\
& -\rho\left(x+\frac{\Delta x}{2}, y+\Delta y, z+\frac{\Delta z}{2}\right) v_y\left(x+\frac{\Delta x}{2}, y+\Delta y, z+\frac{\Delta z}{2}\right) dt\Delta x\Delta z \\
& +\rho\left(x+\frac{\Delta x}{2}, +\frac{\Delta y}{2}, z\right) v_z\left(x+\frac{\Delta x}{2}, +\frac{\Delta y}{2}, z\right) dt\Delta x\Delta y \\
& -\rho\left(x+\frac{\Delta x}{2}, +\frac{\Delta y}{2}, z+\Delta z\right) v_z\left(x+\frac{\Delta x}{2}, +\frac{\Delta y}{2}, z+\Delta z\right) dt\Delta x\Delta y
\end{aligned}$$

Note that the difference of the first two terms can be rewritten:

$$\rho\left(x, y+\frac{\Delta y}{2}, z+\frac{\Delta z}{2}\right) v_x\left(x, y+\frac{\Delta y}{2}, z+\frac{\Delta z}{2}\right) dt\Delta y\Delta z$$
$$-\rho\left(x+\Delta x, y+\frac{\Delta y}{2}, z+\frac{\Delta z}{2}\right) v_x\left(x+\Delta x, y+\frac{\Delta y}{2}, z+\frac{\Delta z}{2}\right) dt\Delta y\Delta z$$
$$=-\frac{\partial\left[\rho(x,y,z,t)v_x(x,y,z,t)\right]}{\partial x} dtdxdydz.$$

We can write similar expressions for the other two pairs of terms to conclude

$$dm=-dtdxdydz\left[\frac{\partial\left[\rho v_x\right]}{\partial x}+\frac{\partial\left[\rho v_y\right]}{\partial y}+\frac{\partial\left[\rho v_z\right]}{\partial z}\right]$$
$$=-dtdxdydz\ \vec{\nabla}\cdot\left(\rho\vec{v}\right).$$

Recall that the mass inside the small box with volume $dV = dxdydz$ is $m=\rho(x,y,z,t)dxdydz.$

Since we're evaluating things for a stationary volume element at the point *x, y, z* with side lengths *dx, dy, dz* we can say that the total time derivative of the mass inside the volume is

$$\frac{dm}{dt}=\left.\frac{d\left[\rho(x,y,z,t)dxdydz\right]}{dt}\right|_{\substack{\text{fixed}\\ x,y,z,dx,dy,dz}}=\frac{\partial\rho(x,y,z,t)}{\partial t}dxdydz$$

because that's what we mean by taking a partial derivative: we hold all the variables except one fixed.

But we also know that $dm = -dtdxdydz\ \vec{\nabla}\cdot\left(\rho\vec{v}\right)$ so it must be true that

$$\frac{dm}{dt} = -\vec{\nabla}\cdot\left(\rho\vec{v}\right)dxdydz.$$

As a result, we can write

$$\frac{dm}{dt} = \frac{\partial\rho\left(x,y,z,t\right)}{\partial t}dxdydz = -\vec{\nabla}\cdot\left(\rho\vec{v}\right)dxdydz$$

so that

$$\frac{\partial\rho\left(x,y,z,t\right)}{\partial t} = -\vec{\nabla}\cdot\left(\rho\vec{v}\right)$$

or

$$\boxed{\frac{\partial\rho\left(x,y,z,t\right)}{\partial t} + \vec{\nabla}\cdot\left(\rho\vec{v}\right) = 0 \qquad \text{(fixed } x,y,z\text{)}}$$

This equation expresses the fact that mass is conserved in our fluid, even though density and velocity of flow can change with time. Keep in mind that I worked this up using a volume element *dV* that was *fixed* in space: with *x, y, z* held constant, our time derivative is a partial derivative $\partial/\partial t$, not a total derivative d/dt.

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Current density and the mass in a macroscopic volume

We can describe the change in mass contained in a macroscopic volume *V* by integrating the above expression:

$$M = \int_V \rho(x,y,z,t)\,dV$$

so

$$\begin{aligned}\frac{dM}{dt} &= \frac{d}{dt}\left[\int_V \rho(x,y,z,t)\,dV\right] \\ &= \int_V \frac{\partial\rho(x,y,z,t)}{\partial t}\,dV = \int_V -\vec{\nabla}\cdot(\rho\vec{v})\,dV\end{aligned}$$

Are you familiar with the divergence theorem? If so, you'll recall that, for any vector field $\vec{J}(x,y,z,t)$,

$$\int_{closed\ surface} \vec{J}(x,y,z,t)\cdot d\vec{A} = \int_{volume\ enclosed} \vec{\nabla}\cdot\vec{J}(x,y,z,t)\,dV\,.$$

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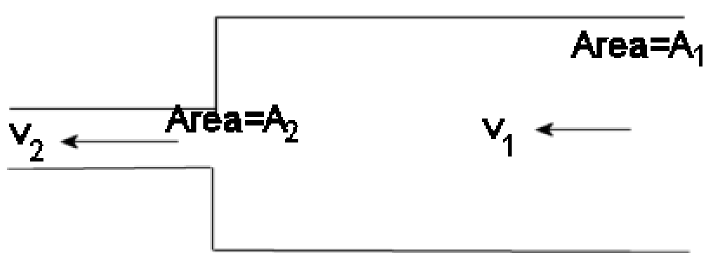
By identifying $\rho\vec{v} \Leftrightarrow \vec{J}$ we can rewrite $\int_V -\vec{\nabla}\cdot(\rho\vec{v})dV$ as

$\int_A -\rho\vec{v}\cdot d\vec{A}$ and conclude that $\frac{dM}{dt} = \int_V -\vec{\nabla}\cdot(\rho\vec{v})dV = \int_A -\rho\vec{v}\cdot d\vec{A}$.

The quantity $\rho\vec{v}$ is the mass *current density* flowing in the fluid.

The integral over a closed surface of the current density flowing through the surface tells us the rate at which mass is entering (or leaving) the volume.

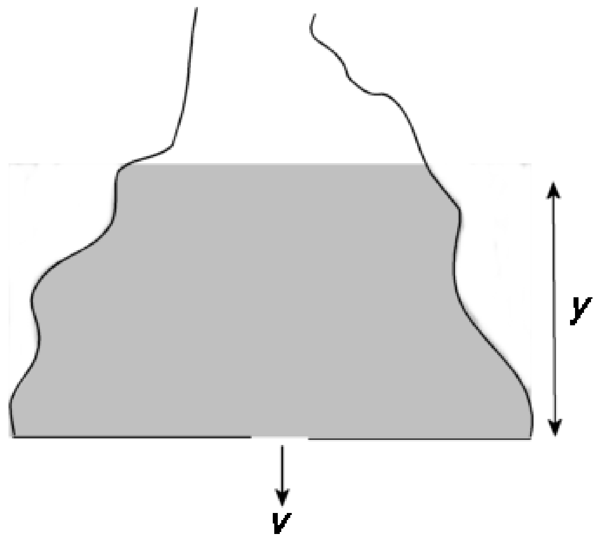
Example 1:



Consider air flowing from a tube with cross-sectional area A_1 into a region with cross-sectional area A_2. In a steady air flow, *dM*/*dt*=0 and so $\rho v_1 A_1 = \rho v_2 A_2$. We have

$$v_2 = \frac{A_1}{A_2} v_1$$

Example 2: Consider water in the following container.



There is a small hole at the bottom of the container and water leaks out from the hole at speed *v*. The water's height *y* decreases.

Mass of water in the container, $M = \rho V$, where *V* is the volume of the water. We have

$$\frac{dM}{dt} = \rho \frac{dV}{dt} = -\rho v A_h ,$$

where A_h is the area of the hole at the bottom. Let *A(y)* be the cross-sectional area of the container at height *y*. Then

$$\frac{dV}{dt} = A(y) \frac{dy}{dt}$$

Hence we have

$$\frac{dy}{dt} = -\frac{A_h}{A(y)} v$$

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Density changes in a "comoving" frame

If we wanted to discuss the rate of change of density of the fluid as we move along with it, we could use the expression for the "convective derivative" from some pages back:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}.$$

As long as we plug in the velocity $\vec{v}$ that corresponds to the (moving) point in the fluid we're observing, we'll learn something useful.

We want to calculate an expression for the total derivative $d\rho/dt$, so let's use the connection between convective and partial derivatives, above:

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \vec{v} \cdot \vec{\nabla}\rho. \qquad [1]$$

We've already figured out how the partial time derivative of ρ (when we're holding *x*, *y*, *z* fixed) behaves:

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \left(\rho \vec{v}\right). \qquad [2]$$

It comes from conservation of mass: if the density is decreasing at a particular point, there has to be an outflow of matter from that point. (The outflow is what the divergence is telling us about.)

Use the result in [2] to replace the first term to the right of the "=" in [1]:

$$\frac{d\rho}{dt} = -\vec{\nabla}\cdot\left(\rho\vec{v}\right) + \vec{v}\cdot\left(\vec{\nabla}\rho\right) \quad [3]$$

or

$$\frac{d\rho}{dt} - \vec{v}\cdot\left(\vec{\nabla}\rho\right) + \vec{\nabla}\cdot\left(\rho\vec{v}\right) = 0\,.$$

Now,

$$\begin{aligned}\vec{\nabla}\cdot\left(\rho\vec{v}\right) &= \frac{\partial}{\partial x}\left(\rho v_x\right) + \frac{\partial}{\partial y}\left(\rho v_y\right) + \frac{\partial}{\partial z}\left(\rho v_z\right) \\ &= \rho\frac{dv_x}{dx} + \left(\frac{d\rho}{dx}\right)v_x + \rho\frac{dv_y}{dy} + \left(\frac{d\rho}{dy}\right)v_y + \rho\frac{dv_z}{dz} + \left(\frac{d\rho}{dz}\right)v_z \\ &= \rho\vec{\nabla}\cdot\vec{v} + \vec{v}\cdot\left(\vec{\nabla}\rho\right)\end{aligned}$$

so I can rewrite $\frac{d\rho}{dt} = -\vec{\nabla}\cdot\left(\rho\vec{v}\right) + \vec{v}\cdot\vec{\nabla}\rho$ (this was equation [3])

as

$$\frac{d\rho}{dt} = -\rho\vec{\nabla}\cdot\vec{v} - \vec{v}\cdot\left(\vec{\nabla}\rho\right) + \vec{v}\cdot\vec{\nabla}\rho$$
$$= -\rho\vec{\nabla}\cdot\vec{v}$$

or

$$\boxed{\frac{d\rho}{dt} + \rho\vec{\nabla}\cdot\vec{v} = 0} \qquad \text{(moving along with fluid)}. \quad [4]$$

That's what we wanted: as we cruise along with the fluid, we'll see the density changing in accord with this equation. The physical meaning is straightforward: if we see atoms in our fluid streaming out from a point (so that the divergence of the velocity is non-zero), we'll expect to see the fluid's density at that point changing with time.

For an incompressible fluid, $\frac{d\rho}{dt} = 0$. So we have $\nabla\cdot\vec{v} = 0$.

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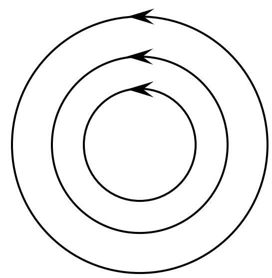
(Ir)rotational flow

Remember about curl? Look over material in the lecture notes on conservative forces for a refresher, if necessary. The vorticity is defined as $\vec{\omega} = \vec{\nabla} \times \vec{v}$. It describes the local spinning motion of fluid.

Let's say the velocity in the fluid near a *vortex* looks like this:



To have something concrete to work with, let's say the velocity at a point (r,θ) is $\vec{v} = v(r)\hat{\theta}$ where $r = 0$ is the center of the vortex.

In Cartesian coordinates, we could write

$$\vec{v} = v\left(\sqrt{x^2 + y^2}\right) \left(\cos\theta\hat{y} - \sin\theta\hat{x}\right).$$

If you look up the form for curl in cylindrical coordinates you'll find that it's

$$\vec{\nabla} \times \vec{A} = \frac{1}{r}\left(\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right)\hat{r} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\theta} + \left(\frac{\partial A_\theta}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \theta} + \frac{A_\theta}{r}\right)\hat{z}.$$

As a result, since $v_\theta = v(r)$, the vorticity is

$$\vec{\omega} = \vec{\nabla} \times \vec{v} = \left[\frac{\partial}{\partial r} v(r) + \frac{v(r)}{r}\right] \hat{z}$$

because $v_r = v_z = 0$.

A circular flow pattern like this has non-zero vorticity.

If the layers are all turning with the same angular velocity, there'll be no "shearing" between adjacent layers. This means that one layer in the flow does not slide past another, and there will be no energy loss associated with viscous drag that one layer will exert on another.

If all layers turn with the same angular velocity Ω, we'll have $v(r) = \Omega r$ which will yield

$$\vec{\omega} = \left[\frac{\partial}{\partial r} v(r) + \frac{v(r)}{r}\right] \hat{z} = 2\Omega \hat{z}$$

independent of r.

If $\vec{\omega}$ depends on r, there'll be shear dislocations of adjacent layers relative to each other, which will lead to energy dissipation in a viscous fluid.

If $\vec{\omega} = 0$, the flow is *irrotational*: a small pinwheel placed in the fluid won't spin.

We can be a little more formal than using the pinwheel analogy by referring to Stokes' theorem:

$$\oint_{loop} \vec{v} \cdot d\vec{s} = \iint_{area} \left(\vec{\nabla} \times \vec{v}\right) \cdot d\vec{A} = \iint_{area} \vec{\omega} \cdot d\vec{A}\,.$$

If the velocity seems to go around in circles so that $\oint_{loop} \vec{v} \cdot d\vec{s} \neq 0$, we'll automatically have non-zero vorticity inside the loop.

It'll also be true that fluid flow with a velocity gradient—for example, having v_x increase with y—can correspond to "rotational flow," even if all molecules in the fluid are traveling in the x direction.

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Equations of motion for an ideal fluid

Let's assume for the moment that we're working with a fluid that has *zero* viscosity, so that one layer of fluid sliding past another layer of fluid *does not* exert a force on the other layer, attempting to drag it along. In the language of fluid mechanics, there is no shear stress exerted by one layer moving across another layer.

Shear stress is defined as the force per unit area that one layer exerts on another layer. "Ideal fluids" do not "support" shear stresses.

The net force acting on a small volume element can come from two sources: a "body force" (for example, gravity) and a pressure gradient that makes the force on one side of the volume element different from the force on the opposite side of the volume element.

Let's say the volume is $dV = dxdydz$ for a small rectangular solid in the fluid. The areas of the six faces are $dxdy$, $dydz$, and so forth.

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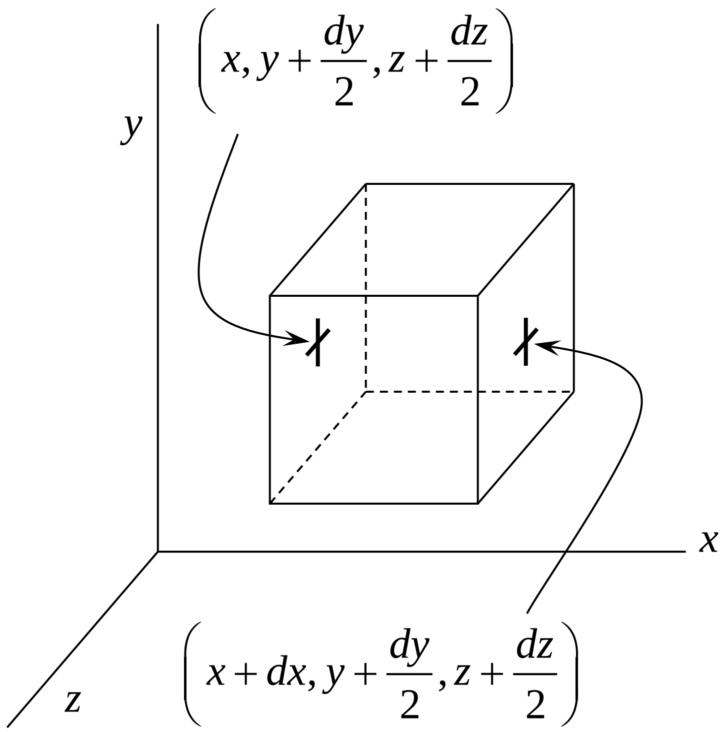

The force associated with pressure on the face centered at

$\left(x, y+\frac{dy}{2}, z+\frac{dz}{2}\right)$ is $P\left(x, y+\frac{dy}{2}, z+\frac{dz}{2}\right)dydz$ since force is the product of pressure and area.

The net force associated with pressure in the x direction is

$$P\left(x, y+\frac{dy}{2}, z+\frac{dz}{2}\right)dydz - P\left(x+dx, y+\frac{dy}{2}, z+\frac{dz}{2}\right)dydz$$
$$= -\frac{\partial P}{\partial x}dxdydz.$$

since the pressure on the right-side face creates a push to the left.

Consequently, we can write the net force, associated with changes in pressure across the volume element, as

$$\begin{aligned}\vec{F}_{gradient} &= \left[-\frac{\partial P}{\partial x}\hat{x} - \frac{\partial P}{\partial y}\hat{\mathrm{y}} - \frac{\partial P}{\partial z}\hat{\mathrm{z}}\right]dxdydz \\ &= \left(-\vec{\nabla}P\right)dV.\end{aligned}$$

If the gravitational force acting on *dV* is to be included, we'll have an additional piece to incorporate which is $m\vec{g} = (\rho dV)\vec{g}$.

The equation of motion for the volume element *dV* is just $\vec{F} = m\vec{a}$, or $\vec{F} = (\rho dV)d\vec{v}/dt$ so

$$(\rho dV)\frac{d\vec{v}}{dt} = \left(-\vec{\nabla}P\right)dV + (\rho dV)\vec{g}$$

Note that $\vec{v}$ refers to the velocity of our tiny volume of fluid, since we're referring to how this velocity changes when there are unequal pressures on opposite sides of the small volume element.

We can rewrite the last equation as $\rho\dfrac{d\vec{v}}{dt} + \vec{\nabla}P = \rho\vec{g}$, or

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$$\frac{d\vec{v}}{dt}+\frac{\vec{\nabla}P}{\rho}=\vec{g}\,.$$

Keep in mind that the pressure P appears in this equation because gradients in pressure will cause unequal forces to be exerted on opposite sides of our tiny volume, causing it to accelerate.

The equation lets us relate density, the pressure gradient, and the acceleration of a packet of fluid as it moves along. It's just telling us that force is mass times acceleration, nothing more. If the fluid is viscous, there'll be additional terms associated with viscous forces acting on our small packet of fluid.

We can write a new version of this describing what happens at a fixed point (as opposed to what happens to a particular group of molecules in the tiny volume that flows from place to place) by using our convective derivative since that'll let us get to $\partial\vec{v}/\partial t$.

Use $\frac{d}{dt}=\frac{\partial}{\partial t}+\vec{v}\cdot\vec{\nabla}$ to rewrite the previous equation as

$$\frac{\partial\vec{v}}{\partial t}+\vec{v}\cdot\vec{\nabla}\vec{v}+\frac{\vec{\nabla}P}{\rho}=\vec{g}.$$

Now we have an equation with a partial derivative. In the case that the flow is steady, so that the behavior of the (moving) fluid at one point in space doesn't change, we'll have $\vec{v}\cdot\vec{\nabla}\vec{v}+\vec{\nabla}P/\rho=\vec{g}$.

Here's what I mean by the curious term $\vec{v}\cdot\vec{\nabla}\vec{v}$ (curious because we appear to be taking the gradient of a vector, instead of a scalar). It is perhaps more clearly expressed as $\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v}$:

$$\begin{aligned}\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} &= \left[v_x\frac{\partial}{\partial x}+v_y\frac{\partial}{\partial y}+v_z\frac{\partial}{\partial z}\right]\vec{v}\\ &= \left[v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right]\hat{x}\\ &\quad+\left[v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right]\hat{y}\\ &\quad+\left[v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right]\hat{z}.\end{aligned}$$

The equation $\dfrac{\partial\vec{v}}{\partial t}+\vec{v}\cdot\vec{\nabla}\vec{v}+\dfrac{\vec{\nabla}P}{\rho}=\vec{g}$ is referred to as Euler's equation of motion for a moving fluid subject to a gravitational force. It is *nonlinear* due to the presence of the $\vec{v}\cdot\vec{\nabla}\vec{v}$ term.

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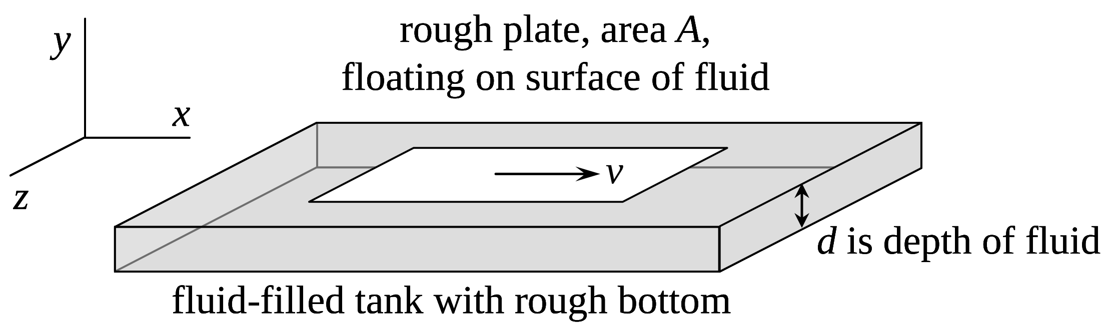
Shear stress in a "Newtonian fluid"

Imagine you drag a sheet of plywood of area A across the floor, using a spring balance to measure the force F necessary to overcome friction. The force per unit area required to move the plywood is F/A, of course; we refer to this as the *stress* caused by the interaction between the plywood and the floor.

Imagine instead that we have a layer of fluid with non-zero viscosity between a (rough) plate of area A and the rough bottom of a large tank, and drag the plate with velocity $\vec{v} = v_{\text{plate}}\hat{x}$ across the top of the fluid, as shown in the figure.



If the bottom of the tank and the surface of the plate that is in contact with the fluid are both sufficiently rough, the layer of fluid in contact with the surfaces should move with the same velocity as the surfaces. The fluid at $y = d$ immediately below the plate will move to the right with velocity $v_{\text{plate}}\hat{x}$ while the fluid at the bottom of the tank will remain at rest.

In a *Newtonian* fluid, the fluid velocity below the moving plate increases linearly from 0 to $v_{\text{plate}}\hat{x}$ as y increases from 0 to d. We can write

$$\frac{\partial v_x(x,y,z)}{\partial y} = \frac{v_{\text{plate}}}{d}.$$

A Newtonian fluid is viscous (otherwise there'd be no shear forces to make the fluid move parallel to the plate), and will exert a stress τ (force per unit area) on the dragged plate that is proportional to how fast we're pulling it, among other things.

(Liquid) water behaves like a Newtonian fluid. Examples of non-Newtonian fluids include Silly Putty and chilled caramel ice cream toppings: stress them hard enough and they'll behave like solids. (Silly Putty will shatter, for example, when the applied force comes from a hammer.)

The thicker the layer of fluid, the less the force needed to drive the plate with constant speed in opposition to the viscous drag exerted by the fluid.

If we can assume that the fluid at the bottom of the tank has zero velocity, while the fluid in contact with the bottom of the plate has the same velocity as the plate, the required force will be proportional to 1/*d* for a Newtonian fluid. As a result, the shear stress on the plate (and on the bottom of the tank) has magnitude

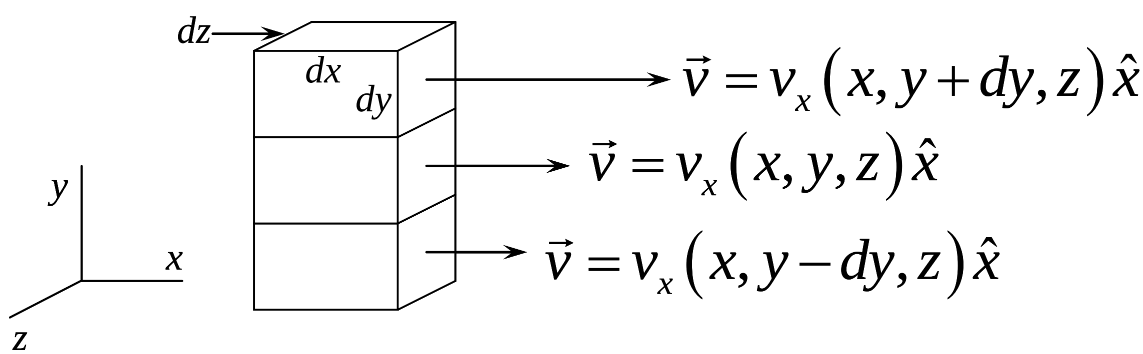
$$\tau \equiv \frac{F}{A} = \mu \frac{v_{\text{plate}}}{d} = \mu \frac{\partial v_x}{\partial y}.$$

The constant μ is an intrinsic property of the fluid, and is called the fluid's *viscosity*.

It's not going to be true in general that the fluid velocity increases linearly with distance transverse to the direction of flow: that's only the case for the example of a plate being dragged above a rough surface.

Imagine we've set up a "steady flow" in a fluid: there's no explicit time dependence to the fluid velocity so we can write it as $\vec{v}(x,y,z)$. Let's look at how viscous forces are exerted on one small volume element by adjacent volume elements. Let's make them all the same size, and assume that the general direction of flow is along *x*, and that the flow speed increases (but not necessarily linearly) with *y*.

Viscous drag from the bottom volume element creates shear stress that tries to slow the middle element down; shear stress from the upper volume element will try to speed the middle element up. The shear stresses will also distort the volume element; we'll look at what's going on at the instant when the collection of molecules in the middle volume happens to form a rectangular solid.



The net (shear) force on the middle element will be

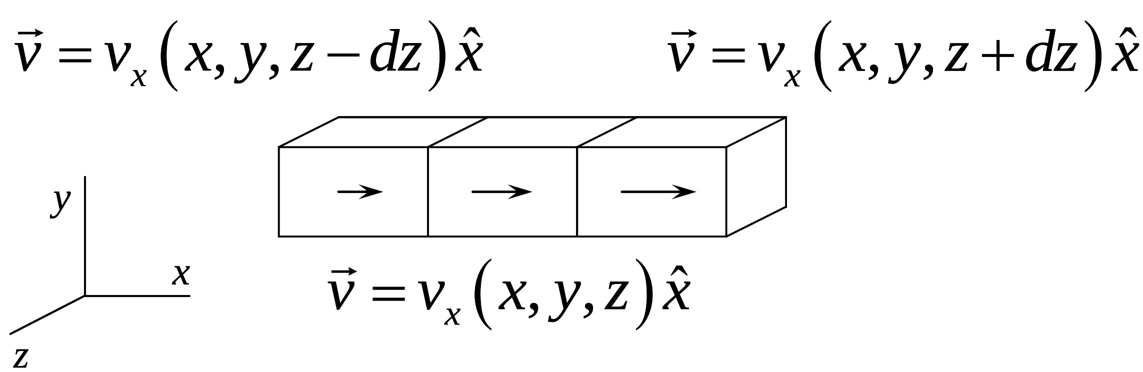
$$F_x = \mu\left(\frac{dv_x}{dy}\bigg|_{\substack{\text{upper}\\\text{surface}}} - \frac{dv_x}{dy}\bigg|_{\substack{\text{lower}\\\text{surface}}}\right)dxdz \approx \mu\left[\frac{d}{dy}\left(\frac{dv_x}{dy}\right)\right]dydxdz$$

$$= \mu\frac{d^2v_x}{dy^2}dxdydz = \mu\frac{d^2v_x}{dy^2}dV.$$

If we allow for a similar velocity gradient in v_x in the z direction, we'll have an additional contribution to the force on our middle volume element. As a result, we expect the force from viscosity on our volume element to be

$$F_x = \mu\left[\frac{d^2v_x}{dy^2} + \frac{d^2v_x}{dz^2}\right]dV.$$

There are also viscous effects that act along the direction of motion of the fluid: if you've ever poured honey into a cup of tea, you've seen this. Even though gravity is pulling honey off your spoon, the viscosity of the honey keeps it from going into freefall.



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The volume element to the left exerts a pull in the negative x direction, the volume to the right a pull in the positive x direction. In a Newtonian fluid, the pull along the direction of motion associated with viscosity (this is distinct from a pull associated with a pressure gradient) will also be proportional to the derivative of the velocity:

$$\frac{F_x}{A} \propto \frac{\partial v_x}{\partial x}.$$

I won't derive it, but the proportionality constant in a Newtonian fluid is the same viscosity constant μ as for shear forces.

The net force on the volume from the non-zero $\partial v_x / \partial x$ is

$$F_x = \mu \left(\left.\frac{dv_x}{dx}\right|_{\substack{\text{right}\\\text{surface}}} - \left.\frac{dv_x}{dx}\right|_{\substack{\text{left}\\\text{surface}}} \right) dydz \approx \mu \left[\frac{d}{dx}\left(\frac{dv_x}{dx} \right) \right] dxdydz$$

$$= \mu \frac{d^2 v_x}{dx^2} dxdydz = \mu \frac{d^2 v_x}{dx^2} dV.$$

As a result, the full expression for the viscous force is nicely symmetric:

$$F_x = \mu \left[\frac{d^2 v_x}{dx^2} + \frac{d^2 v_x}{dy^2} + \frac{d^2 v_x}{dz^2} \right] dV = \left(\mu dV \right) \nabla^2 v_x.$$

Including the effects of possible flow velocities along *y* and *z* gives

$$\vec{F} = \mu dV \left[\hat{x}\nabla^2 v_x + \hat{y}\nabla^2 v_y + \hat{z}\nabla^2 v_z \right] = \left(\mu dV \right) \nabla^2 \vec{v}.$$

Conservation laws

We worked up the equations some pages back to express the fact that mass is conserved in our fluid.

Let's try for a conservation-of-momentum equation now. Start with this equation that I had derived for a non-viscous fluid:

$$\frac{d\vec{v}}{dt} + \frac{\vec{\nabla} P}{\rho} = \vec{g}.$$

This describes the relationship between velocity, pressure, density, and the gravitational acceleration for a point moving along with the fluid, and *not* at a fixed *x, y, z*. That's because we have a total derivative $d\vec{v}/dt$, not a partial derivative, $\partial\vec{v}/\partial t$, as had been used in Euler's equation of motion.

For a *particular* set of molecules in the fluid (we tag them, and keep track of them), the volume they occupy will increase when the density decreases, and decrease when the density increases.

The mass of this particular small set of molecules is *dm*; they occupy volume *dV* and have density ρ. We have $dm = \rho dV$; since we're keeping track of what these particular molecules are doing as they flow along, *dm* will be constant as long as we don't loose track of any of them. As a result, ρdV is constant.

Multiply our equation $\frac{d\vec{v}}{dt} + \frac{\vec{\nabla} P}{\rho} = \vec{g}$ by ρdV to write

$$\rho dV \frac{d\vec{v}}{dt} + \left(\vec{\nabla} P\right) dV = \rho \vec{g} dV .$$

Because ρdV is constant, we can put it inside the time derivative:

$$\frac{d}{dt}\left(\rho \vec{v} dV\right) = \left(\rho \vec{g} - \vec{\nabla} P\right) dV.$$

Note that $\rho \vec{v} dV$ is the momentum carried by the particular set of molecules we're watching. Keep in mind that both ρ and dV change as the fluid flows.

If we integrate over a finite volume, we can write

$$\frac{d}{dt}\left(\int_{\text{volume}} \rho\vec{v}dV\right) = \int_{\text{volume}} \left(\rho\vec{g} - \vec{\nabla}P\right)dV \,.$$

I mentioned the divergence theorem some pages ago:

$$\int_{\textit{closed surface}} \vec{J}\left(x,y,z,t\right)\cdot d\vec{A} = \int_{\textit{volume enclosed}} \vec{\nabla}\cdot\vec{J}\left(x,y,z,t\right)dV \,.$$

There's a generalized version of this that says:

$$\int_{\text{volume}} \vec{\nabla}PdV = \int_{\text{surface}} Pd\vec{A}.$$

As a result, we can rewrite the equation

$$\frac{d}{dt}\left(\int_{\text{volume}} \rho\vec{v}dV\right) = \int_{\text{volume}} \left(\rho\vec{g} - \vec{\nabla}P\right)dV$$

as

$$\frac{d}{dt}\left(\int_{\text{volume}} \rho\vec{v}dV\right) = \int_{\text{volume}} \rho\vec{g}dV - \int_{\text{surface}} Pd\vec{A} \,.$$

$\int_{\text{surface}} P d\vec{A}$ is just the net force acting on a macroscopic volume of fluid associated with changes in pressure over the surface of the fluid.

This equation serves as a statement about momentum conservation in our fluid.

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Hydrostatics

Consider a special case where the fluid is static so that the fluid velocity $\vec{v} = 0$ everywhere. The conservation of momentum equation

$$\frac{d\vec{v}}{dt} + \frac{\vec{\nabla} P}{\rho} = \vec{g}$$

reduces to

$$\vec{\nabla} P = \rho \vec{g}$$

which is the equation of hydrostatic equilibrium. This equation tells us that the pressure gradient is parallel to the direction of gravity $\vec{g}$ and so the surfaces of constant pressure (isobars) are perpendicular to $\vec{g}$. Since $\vec{\nabla} \times \vec{\nabla} P = 0$, taking the curl of the equation of hydrostatic equilibrium yields

$$0 = \vec{\nabla} \times \vec{\nabla} P = \vec{\nabla} \times (\rho \vec{g}) = \vec{\nabla} \rho \times \vec{g}$$

So the density gradient is also parallel to $\vec{g}$. Let $\vec{g} = g\hat{z}$ so that $\hat{z}$ points downward. We have $P = P(z)$ and $\rho = \rho(z)$. That is, the density and pressure only depends on the depth z. The equation of hydrostatic equilibrium becomes

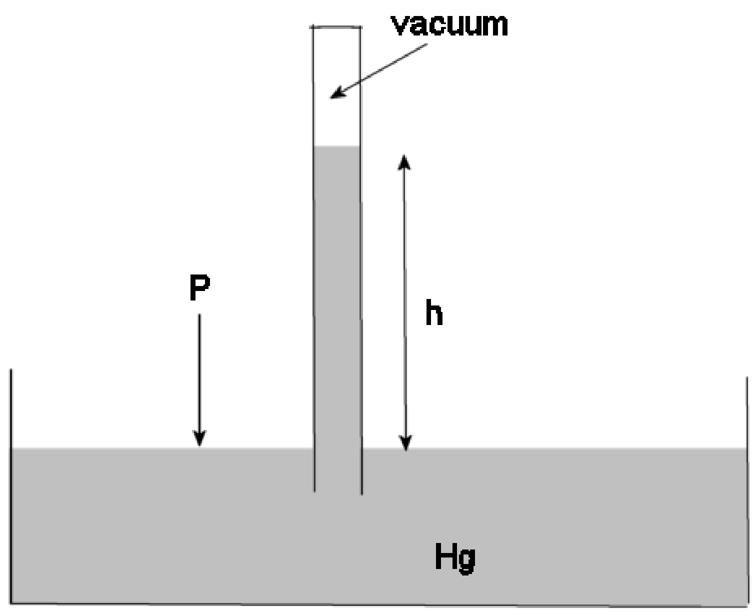
$$\frac{dP}{dz} = \rho g$$

If the fluid density ρ is constant, the equation is easily integrated to give

$$P(z) = P_0 + \rho g z,$$

where P_0 is the pressure at z=0. This equation says that the fluid pressure at a depth z is equal to P_0 plus the total weight of the fluid per unit area above the point.

Mercury Barometer



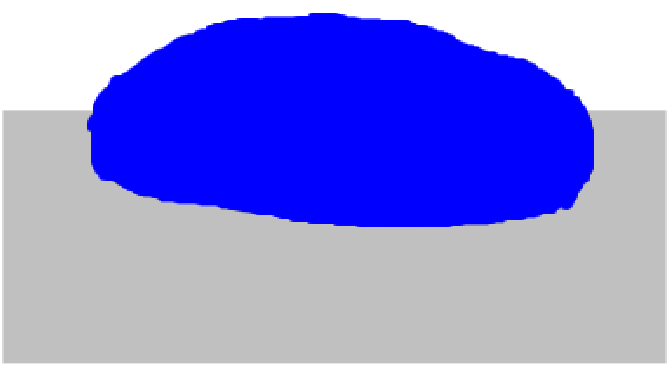
A mercury barometer measures the atmospheric pressure by measuring the height of the mercury column in a glass tube above the mercury-filled basin at the bottom. The pressure is given by

$$P = \rho_{\mathrm{Hg}} g h$$

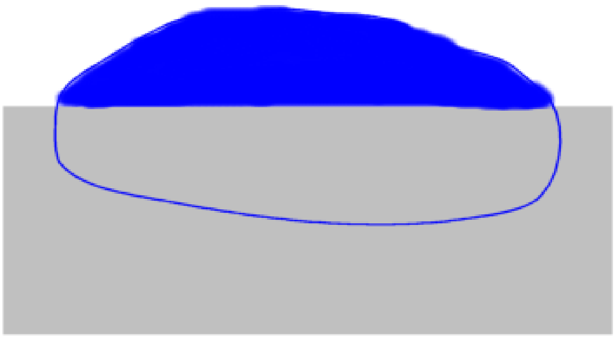
where $\rho_{\mathrm{Hg}} = 13546\mathrm{kg/m}^3$. A millimeter of mercury (mmHg) is formerly defined as the pressure generated by a column of mercury one mm high ($\rho_{\mathrm{Hg}} g \times 1\mathrm{mm} = 132.9$ Pa). Now it's defined as exactly 133.32 Pa. The standard atmosphere pressure is 101kPa, which is about 760 mmHg.

Archimedes' Principle

Consider an object floating stationary in a fluid.



The buoyancy force exerted by the fluid on the object is $\vec{F}_{\mathrm{buoy}} = -\int_{\mathrm{surface}} P d\vec{A}$, where the integral is over the surface of the object immersed in the fluid. Imagine removing the body and replacing it by fluid that has the same density $\rho(z)$ and pressure *P(z)*, at each depth *z*, as the surrounding fluid.



Integrating the equation of hydrostatic equilibrium over the volume of the previously immersed body yields

$$\int_V \vec{\nabla} P\, dV = \int_V \rho \vec{g} dV$$

which can be written as

$$\int_{\text{surface}} P\, d\vec{A} = \int_V \rho \vec{g} dV$$

Hence,

$$\vec{F}_{\text{buoy}} = -\vec{g} \int_V \rho dV = -M_f \vec{g}$$

Here M_f is the mass of the fluid displaced by the body. This is known as *Archimedes' principle*, which states that the upward buoyant force on the body is equal in magnitude to the weight $M_f\, g$ of the displaced fluid.

Earth's Atmosphere

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The variation of Earth's pressure with altitude is closely approximated by the hydrostatic equilibrium.

Let *z* be the upward direction and write $\vec{g} = -g\hat{z}$. The equation of hydrostatic equilibrium becomes

$$\frac{dP}{dz} = -\rho g$$

From the ideal gas law,

$$P = nkT = \frac{\rho}{M} RT \ ,$$

where

$R = N_A k$ = 8.31J/(mol K) is the gas constant,

M = 0.02896 kg/mol is the molar mass of the air (78% N_2, 21% O_2, 0.9% Ar and small amount of other gases).

Combining the two equations yields

$$\frac{dP}{dz} = -\frac{Mg}{RT} P$$

$$\frac{dP}{P} = -\frac{Mg}{RT} dz$$

Integrating both sides gives

$$P(z) = P_0 \exp\left(-\int_0^z \frac{Mg}{RT(z')} dz'\right)$$

Here P_0 is the pressure at z=0. If T=T_0 is constant (isothermal), the above equation becomes

$$P(z) = P_0 \exp\left(-\frac{Mgz}{RT_0}\right) \qquad (isothermal)$$

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The pressure decays exponentially.

A more realistic model assumes that the temperature decreases linearly with height:

$$T = T_0 - Lz\,,$$

where L is called the temperature lapse rate. In this case,

$$\int_0^z \frac{Mg}{RT(z')}dz' = \frac{Mg}{R}\int_0^z \frac{dz'}{T_0 - Lz'} = -\frac{Mg}{RL}\ln\frac{T_0 - Lz}{T_0}$$

and the pressure is

$$P(z) = P_0\left(1 - \frac{Lz}{T_0}\right)^{Mg/RL} \qquad (lapse)$$

Recall that

$$\lim_{k\to\infty}\left(1 + \frac{x}{k}\right)^k = \lim_{k\to\infty}\exp\left[k\ln\left(1 + \frac{x}{k}\right)\right] = \lim_{k\to\infty}\exp\left(k\cdot\frac{x}{k}\right) = e^x$$

It's easy to show that the (*lapse*) equation reduces to the (*isothermal*) equation in the limit $L \to 0$.

The assumption of linearly variation in temperature doesn't hold when at high altitude. A more realistic model is to divide the atmosphere into several layers, and each layer has a different temperature lapse rate. In this model, the pressure in one layer is given by

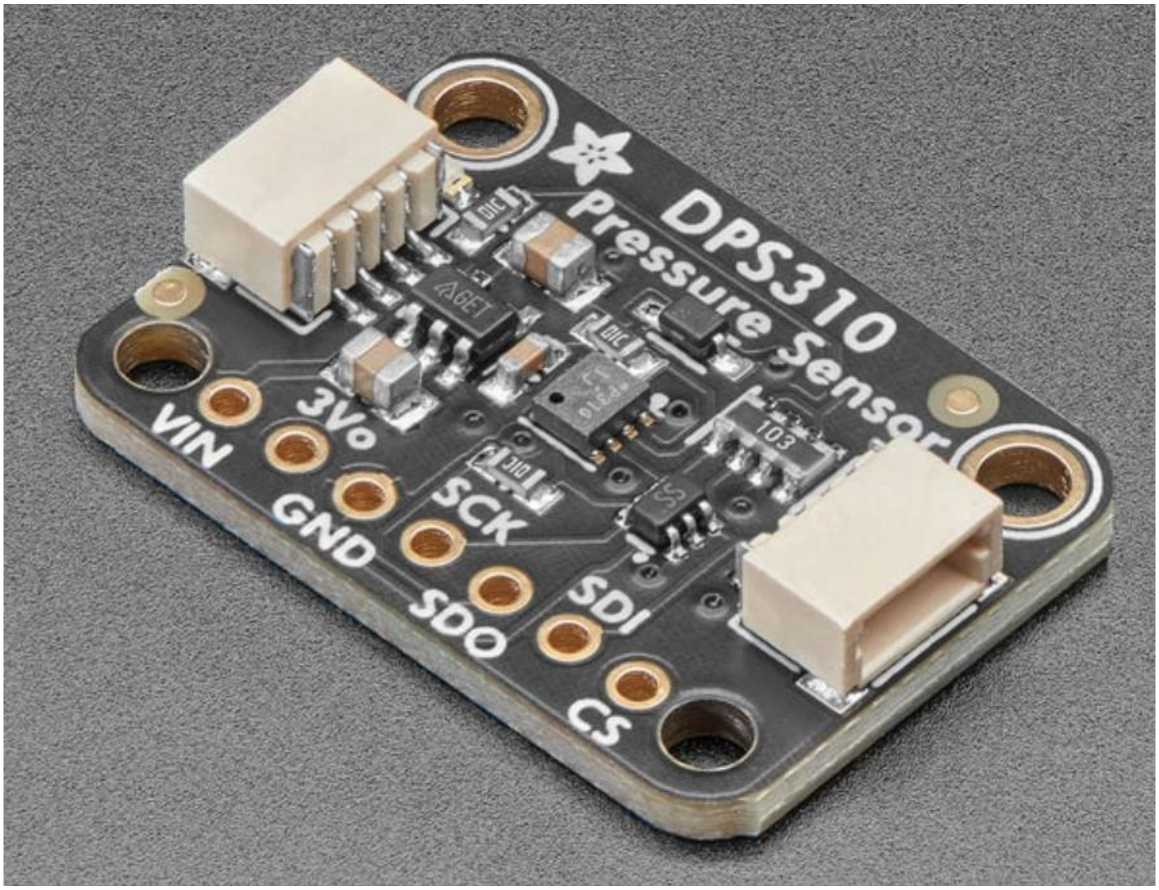
$$P(z) = P_b\left[1 - \frac{L_b(z - z_b)}{T_b}\right]^{Mg/RL_b}$$

Here the subscript *b* ranges from 0 to 6, corresponding to each of the 7 layers of the atmosphere model. The constants are shown in the table below.

Sub-script *b*	**Geopotential height above mean Sea level (z)**		**Static pressure**		**Standard temperature (K)**	**Temperature lapse rate**	
	(m)	**(ft)**	**(Pa)**	**(inHg)**		**(K/m)**	**(K/ft)**
0	0	0	101 325.00	29.92126	288.15	0.0065	0.0019812
1	11 000	36,089	22 632.10	6.683245	216.65	0.0	0.0
2	20 000	65,617	5474.89	1.616734	216.65	-0.001	-0.0003048
3	32 000	104,987	868.02	0.2563258	228.65	-0.0028	-0.00085344
4	47 000	154,199	110.91	0.0327506	270.65	0.0	0.0
5	51 000	167,323	66.94	0.01976704	270.65	0.0028	0.00085344
6	71 000	232,940	3.96	0.00116833	214.65	0.002	0.0006096

Credit: Wikimedia (https://en.wikipedia.org/wiki/Barometric_formula)

Adafruit's DPS 310 Pressure sensor



https://www.adafruit.com/product/4494?gad_source=5

According to Adafruit, their DPS 310 pressure sensor can measure the change in pressure to an accuracy of 0.2 Pa. Recall that

$$\frac{dP}{dz} = -\frac{Mg}{RT}P \quad \Rightarrow \quad \Delta P = -\frac{MgP}{RT}\Delta z$$

Hence a pressure change of $\Delta P = 0.2$ Pa corresponds to a change of height of $\Delta z = 1.7$ cm for P=101 kPa and T=300 K. The pressure sensor can measure altitude to an accuracy of about 2cm.

Energy conservation

Let's develop an energy conservation equation, again assuming the viscosity μ is zero. I'm going to follow the approach in Chapter 12 of the lecture notes by Blandford and Thorne.

Consider a small fluid element occupying a volume V. Let ρ and P be the density and pressure. The mass of this fluid element $m = \rho V$ is constant as we follow its motion. However, its density, pressure and volume can change. According to the first law of thermodynamics,

$$dE = dQ - PdV$$

where E is the internal energy, dQ is the amount of heat flowing into the fluid element. Assume the flow is adiabatic so that there is no heat flow (dQ=0). We have $dE = -PdV$, which means that the increase of internal energy is caused by the compression of the fluid. Let $w = E/m$ be the internal energy per unit mass. Using $V = m/\rho$, we can write the first law as

$$mdw = -Pd\left(\frac{m}{\rho}\right)$$

Dividing the equation by m gives $dw = -Pd(\frac{1}{\rho})$. Hence the rate of change of internal energy per mass is

$$\frac{dw}{dt} = -P\frac{d}{dt}\left(\frac{1}{\rho}\right) = -\frac{d}{dt}\left(\frac{P}{\rho}\right) + \frac{1}{\rho}\frac{dP}{dt}$$

or

$$\frac{d}{dt}\left(w+\frac{P}{\rho}\right)=\frac{1}{\rho}\frac{dP}{dt}=\frac{1}{\rho}\frac{\partial P}{\partial t}+\frac{\vec{v}\cdot\vec{\nabla}P}{\rho} \qquad [1]$$

Recall the equation for conservation of momentum:

$$\frac{d\vec{v}}{dt}=-\frac{\vec{\nabla}P}{\rho}+\vec{g}$$

Taking the dot product of the above equation by $\vec{v}$ gives

$$\vec{v}\cdot\frac{d\vec{v}}{dt}+\frac{\vec{v}\cdot\vec{\nabla}P}{\rho}-\vec{v}\cdot\vec{g}=0 \quad [2]$$

The acceleration of gravity $\vec{g}$ can be written as $\vec{g}=-\vec{\nabla}U$, where U is the gravitational potential. Near Earth's surface, $U=gh$, where h is the height above a reference point. Since gravity on Earth is static, $\frac{\partial U}{\partial t}=0$ and we have

$$\frac{dU}{dt}=\frac{\partial U}{\partial t}+\vec{v}\cdot\vec{\nabla}U=\vec{v}\cdot\vec{\nabla}U=-\vec{v}\cdot\vec{g} \quad [3]$$

Write

$$\vec{v}\cdot\frac{d\vec{v}}{dt}=\frac{d}{dt}\left(\frac{1}{2}\vec{v}\cdot\vec{v}\right)=\frac{d}{dt}\left(\frac{1}{2}v^2\right) \quad [4]$$

Combining equations [1]-[4] gives

$$\frac{d}{dt}\left(\frac{1}{2}v^2+\frac{P}{\rho}+U+w\right)=\frac{1}{\rho}\frac{\partial P}{\partial t}$$

For a steady flow, $\frac{\partial P}{\partial t}=0$ and so

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$$\boxed{\frac{1}{2}v^2+\frac{P}{\rho}+U+w=\text{constant}}$$

For an incompressible fluid (e.g., water), w is constant so we have

$$\frac{1}{2}v^2+\frac{P}{\rho}+U=\text{constant}.$$

This is called Bernoulli's equation; it's why airplanes fly. As v increases, P decreases. Thus lift is generated by airfoils as the greater speed across the curved, upper surface is accompanied by decreased pressure.

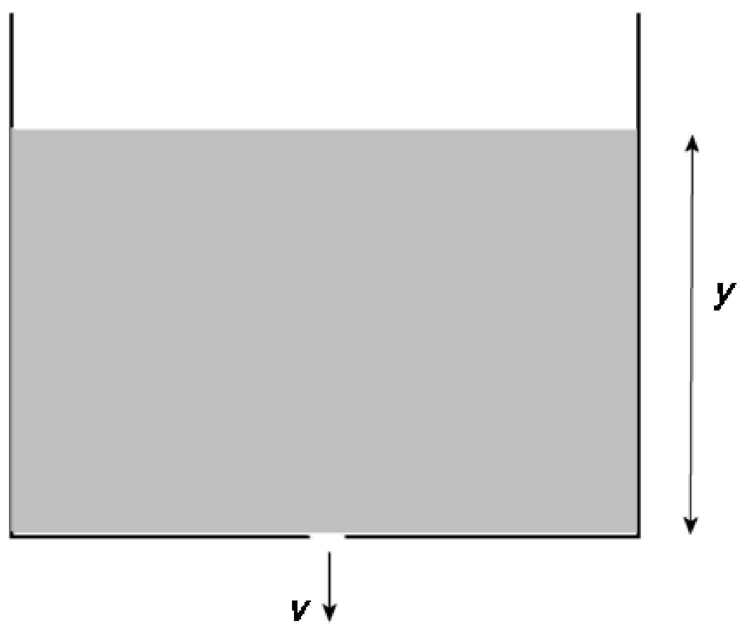
If the fluid flow does not go uphill or downhill so that U is constant, Bernoulli's equation simplifies further:

$$\frac{1}{2}v^2+\frac{P}{\rho}=\text{constant}.$$

Keep in mind how we got to this point: we were investigating how a small volume of fluid accelerated due to the existence of a pressure gradient. Modification of those equations to tell us how the kinetic energy of the small volume of fluid changed thanks to the power applied by the pressure gradient to our volume gave us

Bernoulli's equation. It says, in effect, that the pressure will drop as the fluid expends energy speeding itself up.

Example: Water is flowing out of a rectangular tank from the bottom of a small hole. How long does it take to excavate the water from the tank?



The pressure P is the same at the top of the water level and at the hole. Apply Bernoulli's equation at height y and at the hole:

$$\frac{1}{2}\dot{y}^2 + \frac{P}{\rho} + gy = \frac{1}{2}v^2 + \frac{P}{\rho}$$

$$\Rightarrow \quad v^2 - \dot{y}^2 = 2gy \qquad (1)$$

In a previous example, we find

$$\dot{y} = -\frac{A_h}{A(y)}v = -\frac{A_h}{A}v \quad (2)$$

where A_h is the area of the hole and A the cross-sectional area of the tank. For a rectangular tank, A is independent of y. Combining equations (1) and (2) leads to

$$\left(1-\frac{A_h^2}{A^2}\right)v^2 = 2gy$$

and so

$$v = \sqrt{2gy}\left(1-\frac{A_h^2}{A^2}\right)^{-1/2} \approx \sqrt{2gy} \quad (3)$$

when $A_h \ll A$. This is the speed that a body would acquire in falling freely from a height y. Note that the rate of water flow decreases as the water level y decreases. Combine (2) and (3):

$$\dot{y} = -\frac{A_h}{A}\sqrt{2gy}$$

which can be rewritten as

$$\frac{dy}{\sqrt{y}} = -\frac{A_h}{A}\sqrt{2g}dt$$

Assume that $y=y_0$ at $t=0$. Integrating both sides gives

$$\int_{y_0}^{y}\frac{dy'}{\sqrt{y'}} = -\frac{A_h}{A}\sqrt{2g}\ t$$

$$2\sqrt{y} - 2\sqrt{y_0} = -\frac{A_h}{A}\sqrt{2g}\ t$$

and the water level at time t is

$$y(t) = \left(\sqrt{y_0} - \frac{A_h}{A}\sqrt{\frac{g}{2}}\,t\right)^2$$

Water is excavated from the tank at time T at which $y(T)=0$, or

$$T = \frac{A}{A_h}\sqrt{\frac{2y_0}{g}}$$

Note that $\sqrt{2y_0/g}$ is the time required for an object to fall freely from a height y_0. So T is longer than the free-fall time by a factor of A/A_h.

For $A/A_h = 40$ and $y_0 = 0.3$m, $T \approx$10s. Note that Bernoulli's equation only applies to steady flow. The water flow in the tank is not steady as the flow rate changes with time. However, Bernoulli's equation can still be used if the change is sufficiently slow. The flow is said to be "quasi-steady" in this case. We thus require T to be much longer than the relevant dynamical time scales. There are two dynamical time scales in this problem. The first is associated with pressure, which is characterized by the time it takes for sound to travel a distance y_0. The sound speed in water is about 1500m/s, so y_0/c_s = 0.0002s, which is much shorter than T. The second time scale is associated with gravity, which is characterized by the free-fall time from height y_0. Since T is longer than the free-fall time by the factor $A/A_h = 40$, the quasi-steady approximation is fine and we expect a relative error of about 1/40 = 2.5% in the estimated value of T.

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Viscous Stress tensor

You may have learned about the elastic tensor during an oscillation unit in an intermediate classical mechanics course.

When I described viscous forces that produce shear strains, I concluded that F_x contained contributions from the second derivatives of v_x.

The *stress tensor* is something we can use to streamline some of our notation in fluid dynamics. Here I'm going to follow the approach in Chapter 12 of the lecture notes by Blandford and Thorne.

Let's write the stress tensor $\overleftrightarrow{T}$ in matrix form:

$$\overleftrightarrow{T} = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}$$

The two-sided arrow in the superscript of T indicates that it's a tensor with two indices. It can be shown that the stress tensor must be symmetrical: $T_{ij} = T_{ji}$. Here the indices *i* and *j* run from 1 to 3, with 1 means *x*, 2 means *y* and 3 means *z*. The physical meaning of the stress tensor is that the force acting on a small surface in the fluid $d\vec{A} = \hat{n} dA$ is given by

$$d\vec{F} = \overleftrightarrow{T} \cdot d\vec{A}$$

$$d\vec{F} = dA\left(T_{xx}n_x + T_{xy}n_y + T_{xz}n_z\right)\hat{x}$$

$$+dA\left(T_{yx}n_x + T_{yy}n_y + T_{yz}n_z\right)\hat{y}$$

$$+dA\left(T_{zx}n_x + T_{zy}n_y + T_{zz}n_z\right)\hat{z}$$

Here $\hat{n}$ is the outward unit vector normal to the surface.

Consider a fluid element occupying a certain volume. The total force on this fluid element by the surrounding fluid is

$$\vec{F} = -\int_{\text{surface}} \overleftrightarrow{T} \cdot d\vec{A} = -\int_{\text{volume}} \vec{\nabla} \cdot \overleftrightarrow{T}\, dV$$

where we have used the divergence theorem. The negative sign arises from the fact that for a closed surface, $d\vec{A}$ points out of the fluid element instead of into it. The divergence of the stress tensor is

$$\vec{\nabla} \cdot \overleftrightarrow{T} = \left(\frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{yx}}{\partial y} + \frac{\partial T_{zx}}{\partial z}\right)\hat{x} + \left(\frac{\partial T_{xy}}{\partial x} + \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{zy}}{\partial z}\right)\hat{y}$$

$$+\left(\frac{\partial T_{xz}}{\partial x} + \frac{\partial T_{yz}}{\partial y} + \frac{\partial T_{zz}}{\partial z}\right)\hat{z}$$

which is the negative of force per volume acting on a small fluid element.

The stress tensor of an ideal fluid is $\overleftrightarrow{T} = P\overleftrightarrow{G}$, where P is pressure and $\overleftrightarrow{G}$ is called the metric tensor. In Cartesian coordinates, $\overleftrightarrow{G}$ is represented by a 3×3 identity matrix. In this case, $\overleftrightarrow{T}$ is represented by a diagonal matrix

$$\overleftrightarrow{T} = \begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & P \end{pmatrix}$$

The force acting on a small surface is $d\vec{F} = \overleftrightarrow{T} \cdot d\vec{A} = Pd\vec{A}$. The force is in the direction of $d\vec{A}$ and has equal magnitude in all directions (isotropic). The divergence of $\overleftrightarrow{T}$ is equal to the pressure gradient:

$$\vec{\nabla} \cdot \overleftrightarrow{T} = \left(\frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{yx}}{\partial y} + \frac{\partial T_{zx}}{\partial z}\right)\hat{x} + \left(\frac{\partial T_{xy}}{\partial x} + \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{zy}}{\partial z}\right)\hat{y}$$

$$+ \left(\frac{\partial T_{xz}}{\partial x} + \frac{\partial T_{yz}}{\partial y} + \frac{\partial T_{zz}}{\partial z}\right)\hat{z}$$

$$= \frac{\partial P}{\partial x}\hat{x} + \frac{\partial P}{\partial y}\hat{y} + \frac{\partial P}{\partial z}\hat{z} = \vec{\nabla}P$$

In the presence of viscosity, the stress tensor can be written as the sum of two parts:

$$\overleftrightarrow{T} = P\overleftrightarrow{G} + \overleftrightarrow{\tau}$$

Here $\overleftrightarrow{\tau}$ is called the *viscous stress tensor*. The viscous force acting on a small surface is

$$dF_{\text{vis}} = \overleftrightarrow{\tau} \cdot d\vec{A}$$

The total viscous force on this fluid element by the surrounding fluid is

$$\vec{F}_{\text{vis}} = -\int_{\text{surface}} \overleftrightarrow{\tau} \cdot d\vec{A} = -\int_{\text{volume}} \vec{\nabla} \cdot \overleftrightarrow{\tau}\, dV$$

In the presence of this viscous force per volume, we have to add an extra term $\vec{f}_{\text{vis}} = -\vec{\nabla} \cdot \overleftrightarrow{\tau}$ to the momentum conservation equation:

$$\rho \frac{d\vec{v}}{dt} = \rho \left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v} \right) = -\vec{\nabla} P + \rho \vec{g} - \vec{\nabla} \cdot \overleftrightarrow{\tau}$$

The effect of viscosity is to resist the motion of one layer of fluid slide past another layer. To motivate a model for $\overleftrightarrow{\tau}$ requires us to express mathematically what we mean by “one layer of fluid slide past another layer.”

The motion of fluid is completely described by the fluid velocity field $\vec{v}$. Sliding can occur if neighboring fluid elements move with different velocities. Introduce the velocity gradient tensor

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$$\vec{\nabla}\vec{v} = \begin{pmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_y}{\partial x} & \frac{\partial v_z}{\partial x} \\ \frac{\partial v_x}{\partial y} & \frac{\partial v_y}{\partial y} & \frac{\partial v_z}{\partial y} \\ \frac{\partial v_x}{\partial z} & \frac{\partial v_y}{\partial z} & \frac{\partial v_z}{\partial z} \end{pmatrix}$$

It's useful to introduce a quantity called expansion:

$$\theta = Tr(\vec{\nabla}\vec{v}) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \vec{\nabla} \cdot \vec{v}$$

To understand the physical significance of θ, consider a small fluid element occupying a small volume ΔV. The mass of this fluid element $\Delta m = \rho \Delta V$ is constant as we move with it, but its density and volume may change. So we have

$$0 = \frac{d\Delta m}{dt} = \Delta V \frac{d\rho}{dt} + \rho \frac{d\Delta V}{dt}$$

From the continuity equation, we have $\frac{d\rho}{dt} = -\rho \vec{\nabla} \cdot \vec{v} = -\rho\theta$.

Substituting this to the above equation gives

$$\theta = \frac{1}{\Delta V}\frac{d\Delta V}{dt}$$

Thus, θ is the fractional rate of increase of fluid element's volume.

Next, we introduce the rate of shear tensor $\overleftrightarrow{\sigma}$ and rate of rotation tensor $\overleftrightarrow{r}$ whose components are defined as

$$\sigma_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) - \frac{1}{3}\theta\delta_{ij}$$

$$r_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)$$

Again the indices *i* and *j* run from 1 to 3 (1 means *x*, 2 means *y*, 3 means *z*, and $x_1 = x, x_2 = y, x_3 = z$). The Kronecker delta function is $\delta_{ij} = 1$ if $i = j$ and 0 if $i \neq j$. The rate of shear tensor $\overleftrightarrow{\sigma}$ is symmetry and trace-free (i.e. $\sigma_{ij} = \sigma_{ji}$ and $Tr(\overleftrightarrow{\sigma}) = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 0$). The rate of rotation tensor $\overleftrightarrow{r}$ is anti-symmetry ($r_{ij} = -r_{ji}$) and trace-free. It's easy to show that

$$r_{xy} = -r_{yx} = \frac{1}{2}\omega_z, \qquad r_{yz} = -r_{zy} = \frac{1}{2}\omega_x, \quad r_{zx} = -r_{xz} = \frac{1}{2}\omega_y$$

and the diagonal terms vanish: $r_{xx} = r_{yy} = r_{zz} = 0$.

Here $\vec{\omega} = \vec{\nabla} \times \vec{v}$ is the vorticity we introduced earlier. Physically, $\overleftrightarrow{r}$ describes a rotational motion of the fluid; $\overleftrightarrow{\sigma}$ describes the shear motion of the fluid – deformation that preserves fluid's volume. It's easy to show that the velocity gradient tensor can be decomposed as

$$\left(\vec{\nabla}\vec{v}\right)_{ij} = \frac{1}{3}\theta\delta_{ij} + \sigma_{ij} + r_{ij}$$

The first term corresponds to the expansion and contraction of a fluid, the second term describes the shear motion and the third term describes the rotational motion. It's the shear motion (second term) that causes one layer of a fluid sliding past another layer.

Thus a simple model of the viscous stress tensor is to assume that τ_{ij} is linear to the rate of deformation:

$$\tau_{ij} = -\zeta\theta\delta_{ij} - 2\mu\sigma_{ij}$$

where ζ and μ are called the coefficients of bulk and shear viscosity, respectively. The negative sign is to make viscosity oppose to the motion. In particular, the *bulk viscosity* $-\zeta\theta\delta_{ij}$ resists the fluid's expansion and contraction, and *shear viscosity* $-2\mu\sigma_{ij}$ resists the fluid's shear motion. In general, the bulk viscosity is much smaller than the shear viscosity and is often ignored.

Including viscosity; the Navier-Stokes equations

Our momentum conservation equation in the presence of viscosity is

$$\boxed{\rho\frac{d\vec{v}}{dt} = \rho\left(\frac{\partial\vec{v}}{\partial t} + \vec{v}\cdot\vec{\nabla}\vec{v}\right) = -\vec{\nabla}P + \rho\vec{g} - \vec{\nabla}\cdot\overleftrightarrow{\tau}}$$

This is called the *Navier-Stokes Equation.* Sometimes the "continuity equation" $\frac{\partial\rho(x,y,z,t)}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0$ is included as another of the Navier-Stokes equations.

For an incompressible fluid like water, $\theta = \vec{\nabla}\cdot\vec{v} = 0$. Hence

$$\tau_{ij} = -\mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

Assuming μ is constant, we have

$$\vec{\nabla}\cdot\overleftrightarrow{\tau} = \sum_{1=1}^{3}\frac{\partial}{\partial x_i}\left(\sum_{j=1}^{3}\tau_{ij}\hat{x}_j\right) = -\,\mu\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\frac{\partial^2 v_i}{\partial x_i \partial x_j} + \frac{\partial^2 v_j}{\partial x_i^2}\right)\hat{x}_j$$

The first term is

$$-\mu\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial^2 v_i}{\partial x_i \partial x_j}\hat{x}_j = -\mu\sum_{j=1}^{3}\hat{x}_j\frac{\partial}{\partial x_j}\sum_{i=1}^{3}\frac{\partial v_i}{\partial x_i} = -\mu\vec{\nabla}\left(\vec{\nabla}\cdot\vec{v}\right) = 0$$

for incompressible fluid. The second term is

$$-\mu\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial^2 v_j}{\partial x_i^2}\hat{x}_j = -\mu\sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^2}\sum_{j=1}^{3}v_j\hat{x}_j = -\mu\sum_{i=1}^{3}\frac{\partial^2\vec{v}}{\partial x_i^2} = -\,\mu\nabla^2\vec{v}$$

Hence the viscous force per unit volume is

$$\vec{f}_{\text{vis}} = -\vec{\nabla}\cdot\overleftrightarrow{\tau} = \mu\nabla^2\vec{v}$$

for incompressible fluid.

Water flowing through a long, cylindrical pipe

This is a good example of the power of the Navier-Stokes equations. I'm going to follow the development of the subject in J.M. McDonough's lecture notes, referenced at the beginning of this unit.

Something I forgot to mention: to good accuracy, the layer of a viscous fluid that is in contact with a surface is *always* moving at exactly the same velocity as the surface. This is called the "no-slip condition."

We need to rewrite the Navier-Stokes equations in cylindrical coordinates. That's not a big deal; looking it up (rather than grinding it out myself) yields this:

Continuity (conservation of mass) equation:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial\left(\rho r v_r\right)}{\partial r} + \frac{1}{r}\frac{\partial\left(\rho v_\theta\right)}{\partial \theta} + \frac{\partial\left(\rho v_z\right)}{\partial z} = 0$$

Equation of motion for r component of momentum:

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) =$$

$$-\frac{\partial P}{\partial r} - \left(\frac{1}{r}\frac{\partial\left(r\tau_{rr}\right)}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} - \frac{\tau_{\theta\theta}}{r} + \frac{\partial \tau_{rz}}{\partial z}\right) + F_r$$

Equation of motion for *z* component of momentum:

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)=$$

$$-\frac{\partial P}{\partial z}-\left(\frac{1}{r}\frac{\partial\left(r\tau_{rz}\right)}{\partial r}+\frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta}+\frac{\partial \tau_{zz}}{\partial z}\right)+F_z$$

For a Newtonian incompressible fluid, the momentum equations reduce to the following:

Equation of motion for *r* component of momentum:

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)=$$

$$-\frac{\partial P}{\partial r}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\left(rv_r\right)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial^2 v_r}{\partial z^2}\right]+F_r$$

Equation of motion for *z* component of momentum:

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)=$$

$$-\frac{\partial P}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right]+F_z$$

Due to the cylindrical symmetry of the pipe there cannot be a θ dependence to anything. If there are no gravitational effects, F_r and F_z are zero.

If we look at the fluid far enough from the entrance of the pipe so that we've arrived at a place where the flow is steady, v_r must be zero since the fluid is not passing out of the walls of the pipe. Naturally, v_θ is also zero. Further, the density has no explicit time dependence. As a result, the continuity equation becomes

$$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial(\rho r v_r)}{\partial r}+\frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial\theta}+\frac{\partial(\rho v_z)}{\partial z}=\frac{\partial(\rho v_z)}{\partial z}=\rho\frac{\partial v_z}{\partial z}=0$$

so that v_z can only depend on r, but not on z.

The r momentum equation becomes $\partial P/\partial r=0$ so that the pressure is independent of r, and can only depend on z: $P(z)$.

The z momentum equation becomes

$$0=-\frac{\partial P(z)}{\partial z}+\mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z(r)}{\partial r}\right)$$

or

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$$\frac{\partial P(z)}{\partial z}=\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z(r)}{\partial r}\right).$$

Note that the left side is only a function of *z*, while the right side is only a function of *r*. This can only be true if both sides are *constant*. There is a pressure drop in the pipe, of course, so let's identify the constant as $\Delta P/L$ for a total pipe length of *L*. Note that ΔP is negative: it's final pressure minus initial pressure.

Solve for v_z now, replacing our partial derivatives with total derivatives to be able to do the integrals:

$$\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z(r)}{\partial r}\right)=\frac{\Delta P}{L}\Rightarrow d\left(r\frac{dv_z(r)}{dr}\right)=\frac{\Delta P}{\mu L}rdr\,;$$

Integrate:

$$\int d\left(r\frac{dv_z(r)}{dr}\right)=\int\frac{\Delta P}{\mu L}rdr$$

$$r\frac{dv_z(r)}{dr}=\frac{\Delta P}{2\mu L}r^2+C_1$$

$$\int d\left[v_z(r)\right]=\int\left[\frac{\Delta P}{2\mu L}r+\frac{C_1}{r}\right]dr$$

or

$$v_z(r)=\frac{\Delta P}{4\mu L}r^2+C_1\ln(r)+C_2.$$

Two boundary conditions on v_z: it must be finite at $r = 0$, and it must be zero at the walls of the pipe, where $r = R$. The first requires $C_1 = 0$; the second that $C_2 = -\Delta P R^2/4\mu L$. As a result,

$$v_z(r) = \frac{|\Delta P|}{4\mu L} R^2 \left(1 - \frac{r^2}{R^2}\right).$$

We are neglecting messy things like turbulence!

The average flow velocity in the pipe is this:

$$\langle v_z \rangle = \frac{1}{\pi R^2} \int_0^R \int_0^{2\pi} \frac{|\Delta P|}{4\mu L} R^2 \left(1 - \frac{r^2}{R^2}\right) r\,d\theta\,dr$$

$$= \frac{|\Delta P|}{2\mu L} \int_0^R \left(r - \frac{r^3}{R^2}\right) dr = \frac{|\Delta P|}{2\mu L} \left(\frac{r^2}{2} - \frac{r^4}{4R^2}\right)\Bigg|_0^R = \frac{|\Delta P| R^2}{8\mu L}.$$

The volume of fluid passing through the pipe is just the average flow velocity multiplied by the cross sectional area of the pipe:

$$\text{flow rate} = \frac{\pi |\Delta P| R^4}{8\mu L}.$$

Summary

These are the master equations for fluid dynamics, made with certain simplifying assumptions such as constant, homogeneous viscosity that is independent of flow velocity, pressure, temperature, and so forth.

Conservation of mass ("Continuity Equation")

$$\frac{\partial\rho(x,y,z,t)}{\partial t}+\vec{\nabla}\cdot(\rho\vec{v})=0 \qquad \text{(fixed } x,y,z\text{)}$$

Conservation of momentum ("Navier-Stokes Equation")

$$\rho\left(\frac{\partial\vec{v}}{\partial t}+\vec{v}\cdot\vec{\nabla}\vec{v}\right)=-\vec{\nabla}P+\rho\vec{g}-\vec{\nabla}\cdot\overleftrightarrow{\tau}$$

(Newtonian incompressible fluid)

Conservation of Energy ("Bernoulli's Equation")

$$\frac{1}{2}v^2+\frac{P}{\rho}+U=\text{constant} \qquad \text{(incompressible, non-viscous fluid)}$$

The Navier-Stokes equations are nonlinear, and capable of producing the chaotic behavior—turbulence—that is observed in many fluid systems.

There are loads of concepts in fluid dynamics that I have omitted—Mach number, Reynolds number, turbulence, supersonic flow, and so forth.

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