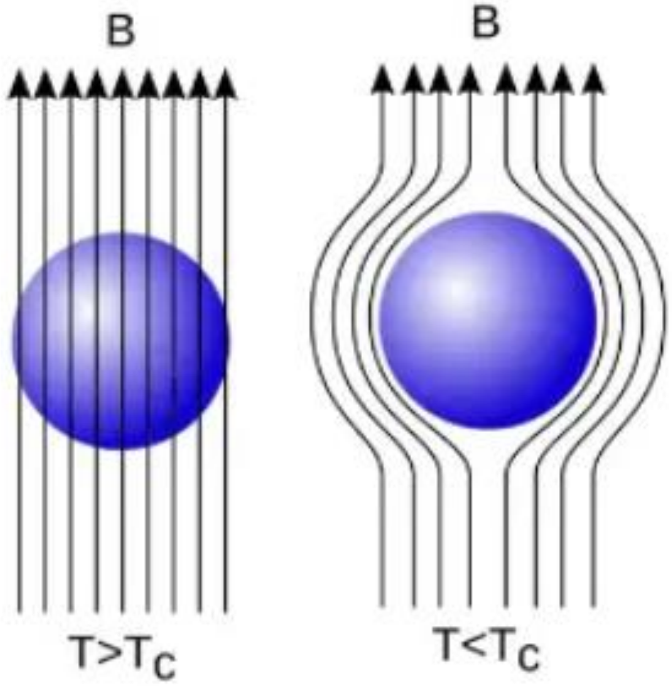
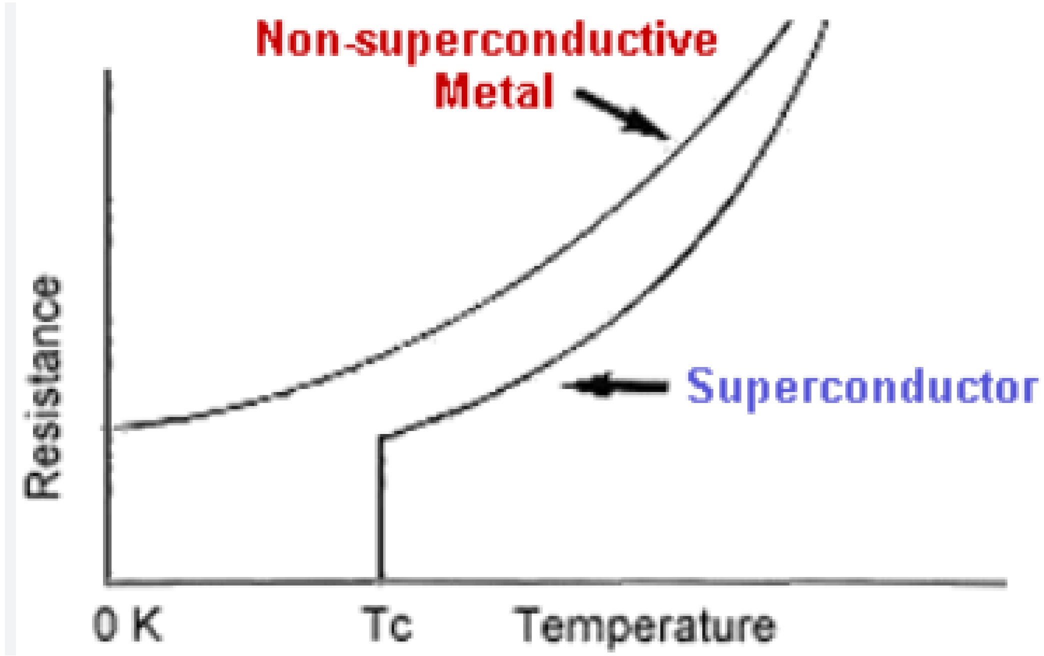
Introduction to theoretical models of superconductivity

Two key features of superconductrs: Superconductivity is based on two basic phenomena. The most important key property of superconductivity is the Meissner effect, which is the expulsion of magnetic fields from the space filled by a superconducting material. The second important signature of superconductors is their zero resistance. Our goal is to learn some simple mathematical models used to describe these and related phenomena.



Meissner effect



Zero resistance at low temperature

https://en.wikipedia.org/wiki/Meissner_effect

https://hst-archive.web.cern.ch/archiv/HST2001/accelerators/superconductivity/superconductivity.htm

Superconducting Condensate: The basic idea is to treat superconductors as quantum systems in which all electrons in the sample can be described by the same wave function, which is analogous to the wave function of just one electron. Such behavior is called coherent behavior. (A more familiar example would be a laser, in which case all photons emitted by the laser are described by the same wave function, with the same frequency, the same wavelength and the same phase.) For a device, to become superconducting, it is sufficient to have a macroscopic fraction of all its electrons to behave coherently. For example, at higher temperatures the fraction of the electrons participating in the superconducting condensate wave function diminishes and actually goes to zero at T=Tc.

Microscopic theory was developed by Bardeen Cooper and Schrieffer (BSC). The BCS theory is quite complicated and so it is beyond the scope of this course. Fortunately, it is possible to take a simplified approach and use a simple model, analogous to the Schrodinger equation, to describe superconductors, at least approximately.

This is how the BCS wave function is constructed:

BCS ground state: $$|\Psi_G\rangle = \prod_{k_1,\ldots,k_m} \left(u_k + v_k c^+_{k\uparrow} c^+_{-k\downarrow}\right)|0\rangle$$

$|u_k|^2 + |v_k|^2 = 1$ Eigenstate of θ so N is indeterminate

This formula is presented as an illustration only.

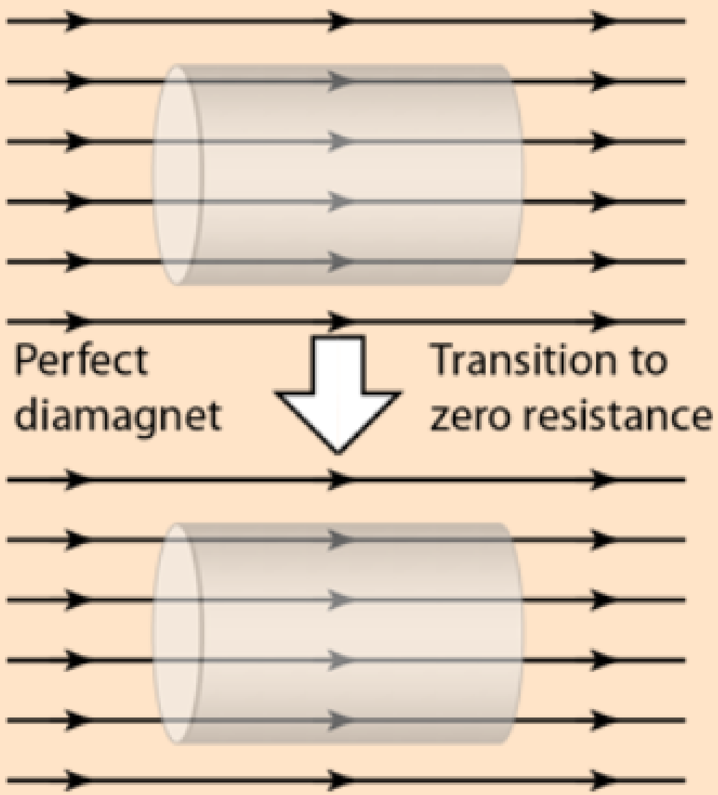
The Meissner Effect

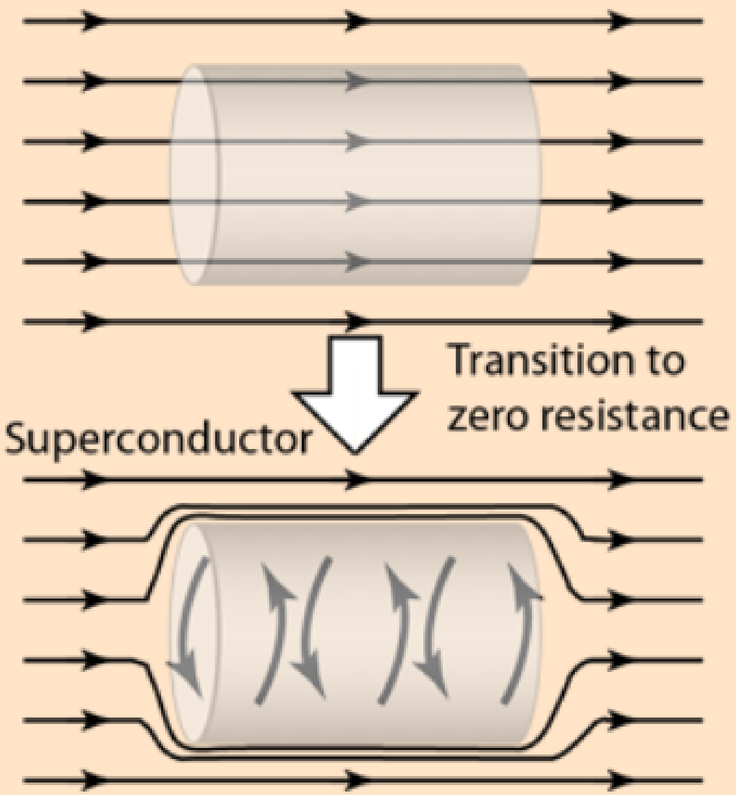
Perfect conductor

If a conductor already had a steady magnetic field through it and was then cooled through the transition to a zero resistance state, becoming a **perfect conductor** the magnetic field would be expected to stay the same.

Superconductor

Remarkably, the magnetic behavior of a superconductor is distinct from perfect ~~diamagnetism~~ conductor. It will actively exclude any magnetic field present when it makes the phase change to the superconducting state.





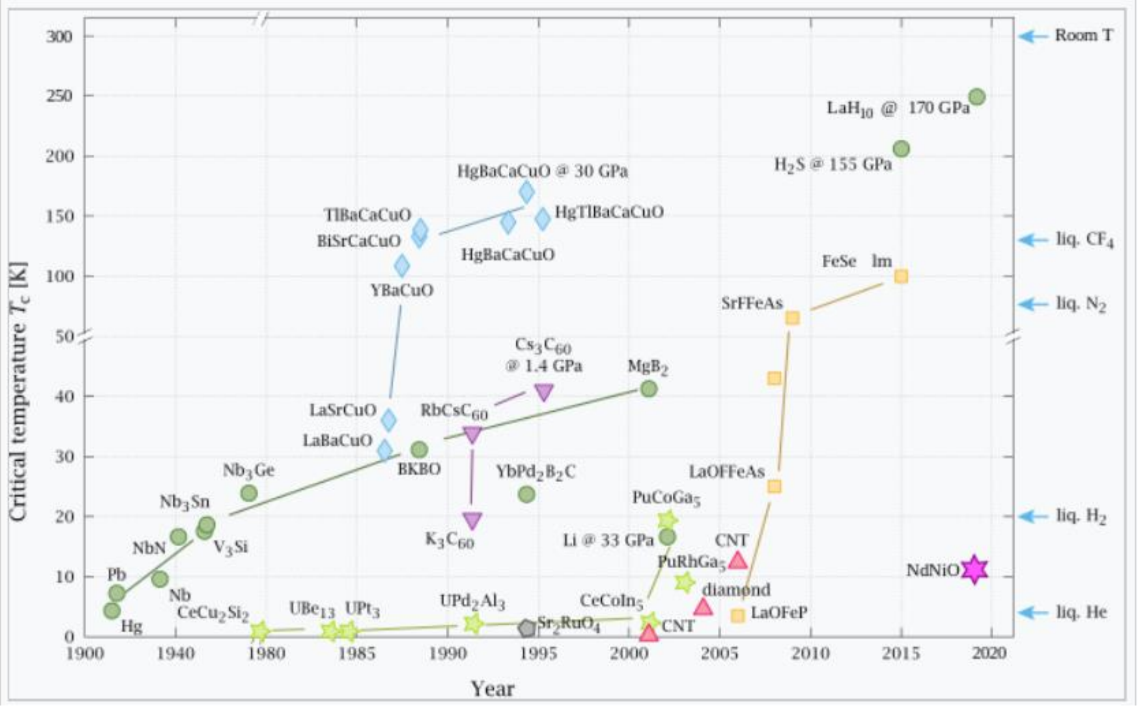
http://hyperphysics.phy-astr.gsu.edu/hbase/Solids/meis.html

Magnetic field penetration depth and coherence length

In superconductors, the magnetic field penetration depth or London penetration depth (sometimes denoted as λ_L) characterizes the depth to which an external magnetic field penetrates the superconductor. Magnetic field decays exponentially, as a function of distance from the surface. The penetration length, λ_L, is chosen such that the magnetic field at this depth equals to B_0/e , where e=2.718 and B_0 is the magnetic field at the surface of the superconductor. Typical values of λ_L range between 50 to 500 nm, as the table below illustrates.

Material	Coherence length ξ_0(nm)	London penetration depth λ_L(nm)	Ratio λ_L/ξ_0
Sn	230	34	0.16
Al	1600	16	0.010
Pb	83	37	0.45
Cd	760	110	0.14
Nb	38	39	1.02

Data attributed to R. Meservey and B. B. Schwartz.

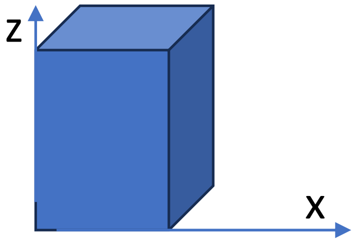


https://en.wikipedia.org/wiki/High-temperature_superconductivity

Type II superconductors are those in which lambda>ksi. Such superconductors support vortices.

The London penetration depth results from considering the London equation and Ampère's circuital law. To make this discussion quantitative, consider a superconducting half-space, i.e a superconducting for x>0, and vacuum for x<0. An external magnetic field $\mathbf{B}_0$ applied along *z* direction in the empty space *x*<0, then inside the superconductor the magnetic field is given by:

$$B(x) = B_0 \exp\left(-\frac{x}{\lambda_L}\right)$$



Here x is the depth inside the superconductor. Here λ_L is the distance or the depth into the superconductor in which the magnetic field becomes *e* times weaker. Here *e*=2.718.

The **London penetration depth**, λ_L, is derived for clean and pure superconductors, without any crystal lattice defects or disorder. The result is: $\lambda_L = \sqrt{\frac{m}{\mu_0 n_s e^2}}$

$$\mu_0 = 4\pi \times 10^{-7}\ \text{H/m}$$

Value of μ_0	Unit
$1.256\,637\,062\,12(19) \times 10^{-6}$	$N \cdot A^{-2}$

Here *m* is the mass of the electrons in the superconductor, *e* is the electronic charge, and n_s is the number density of superconducting electrons in the given superconducting material. The number density, *n*, is also called superfluid density. The electron density for Al, for example, is 1.8×10^{29} m^{-3}.

Maxwell' equations

(we will need some of them to derive lambda)

Maxwell's equations, are a set of coupled partial differential equations that, together with the Lorentz force law, form the foundation of classical electromagnetism, classical optics, and electric circuits. The equations provide a mathematical model for electric, optical, and radio technologies, such as power generation, electric motors, wireless communication, lenses, radar, etc. They describe how electric and magnetic fields are generated by charges, currents, and changes of the fields.

To explain the Meissner effect we will use London model.

It will involve:

1. Newton equation
2. Maxwell equations →

Name	Differential equations	
Gauss's law	$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$	MEq1
Gauss's law for magnetism	$\nabla \cdot \mathbf{B} = 0$	MEq2
Maxwell–Faraday equation (Faraday's law of induction)	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	MEq3
Ampère's circuital law (with Maxwell's addition)	$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$	MEq4

Formulation in SI units convention

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Derivation of London formulas for the magnetic field penetration into a superconductor

Ordinary metals, which are not superconducting, are usually called "normal metals". *Normal metals* behave according to Ohm's law, which states that current is proportional to electric field, *E*.
The normal ("normal"= "not superconducting metal") current density is

$$j_n = \sigma E,$$

where σ is the normal metal conductivity.

However, such a linear relationship is impossible in a superconductor for, almost by definition, the electrons in a superconductor flow with no resistance whatsoever. So the electrons have to be treated as free particles, under the influence of a uniform external electric field. Free particles are described by the Newton equation:

$$m(\mathrm{d}v/\mathrm{d}t) = F = eE \qquad or \qquad \mathrm{d}v/\mathrm{d}t = eE/m \qquad (1)$$

where d*v*/d*t* is the time derivative of the velocity, i.e., the acceleration, *F* is the force acting on each electron. The force, according to the rules of the Coulomb interaction, is given by the charge, *e*, multiplied by the electric field, *E*.

Next step, we need to express the superconducting current ("supercurrent"), j_s, through the density of the superconducting electrons, n_s.

$$j_s = evn_s \qquad (2)$$

If we now combine Eq.1 and Eq.2 we get the first equation which governs the dynamics of supercurrent:

$$(\partial j_s / \partial t) = Ee^2 n_s/\mathrm{m} \qquad (3)$$

Derivation of London formulas for the magnetic field penetration into superconductor

Now we derive another equation, which relates the current and the magnetic field. For this we take curl of the Eq.3:

$$(\partial/\partial t)(\mathrm{curl}\,\mathbf{j}_s) = e^2 n_s(\mathrm{curl}\,\mathbf{E})/m$$

Combine this with the Maxwell equation MEq3 and get:

$$(\partial/\partial t)(\mathrm{curl}\,\mathbf{j}_s) = -e^2 n_s(\partial \mathbf{B}/\partial t)/m$$

Note: Bold letters represent vector physical quantities.

We can now rewrite as follows:

$$(\partial/\partial t)(\mathrm{curl}\,\mathbf{j}_s + e^2 n_s \mathbf{B}/m) = 0$$

Therefore $\mathrm{curl}\,\mathbf{j}_s + e^2 n_s \mathbf{B}/m = \mathrm{constant}$

This constant might not be zero in normal metals. But it is zero in superconductors. Taking this constant to zero means that if there is some non-zero magnetic field then there is some non-zero current. (Fundamentally it is related to the fact that quantum operator of the velocity contains vector-potential.) Thus:

$$\mathrm{curl}\,\mathbf{j}_s + e^2 n_s \mathbf{B}/m = 0 \qquad (4)$$

And this is the second equation which describes vortex-free regime in superconductors. This equation can be presented in a different format if we remember that **B**=curl**A**, where **A** is the vector-potential. Then we get:

$\mathrm{curl}\,(\mathbf{j}_s + e^2 n_s \mathbf{A}/\mathrm{m}) = 0$ or

$$\mathbf{j}_s = -e^2 n_s \mathbf{A}/m \qquad (5)$$

Or, for the superfluid velocity we have:

$$\mathbf{v}_s = -e\,\mathbf{A}/m \qquad (6)$$

This formulas (5 and 6) are applicable if the phase of the superconducting wavefunction is constant everywhere. In the equations 5 and 6 the gauge must be fixed as explained in the next slide.

Gauge freedom

The "field strength" physically measurable variables can be expressed in terms of the electric scalar potential φ and the magnetic vector potential A through the relations: .

$$\mathbf{E} = -\nabla\varphi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}.$$

If the transformation

$$\mathbf{A} \to \mathbf{A} + \nabla\psi \qquad (1)$$

is made, then **B** remains unchanged, since (with the identity $\nabla \times \nabla\psi = 0$)

$$\mathbf{B} = \nabla \times (\mathbf{A} + \nabla\psi) = \nabla \times \mathbf{A}.$$

However, this transformation changes **E** according to

$$\mathbf{E} = -\nabla\varphi - \frac{\partial \mathbf{A}}{\partial t} - \nabla\frac{\partial \psi}{\partial t} = -\nabla\left(\varphi + \frac{\partial \psi}{\partial t}\right) - \frac{\partial \mathbf{A}}{\partial t}.$$

If another change

$$\varphi \to \varphi - \frac{\partial \psi}{\partial t} \qquad (2)$$

is made then **E** also remains the same. Hence, the **E** and **B** fields are unchanged if one takes any function $\psi(\mathbf{r}, t)$ and simultaneously transforms **A** and φ via the transformations (1) and (2).

Choice of gauge

Consider again Eq.5: $\mathbf{j}_s = - e^2 n_s \mathbf{A}/m$

The current density is a well defined measurable physical quantity. The vector potential is not. It can be changed by a adding a gradient of any smooth scalar function because the magnetic field is the curl of the vector potential.

Thus, for the London equation to hold the gauge fixed to the "London gauge“, i.e., the vector potential obeys the following requirements, ensuring that it can be interpreted as a current density:

$\nabla \cdot \mathbf{A} = 0$

$\mathbf{A} = 0$ in the superconductor bulk,

$\mathbf{A} \cdot \mathbf{n} = 0$ where $\mathbf{n}$ is the normal vector at the surface of the superconductor.

The first requirement, also known as Coulomb gauge condition, leads to the constant superconducting electron density, n_s=const, as expected from the continuity equation. The second requirement is consistent with the fact that supercurrent flows only near the superconductor surface. The third requirement ensures no accumulation of superconducting electrons on the surface. These requirements do away with all gauge freedom and uniquely determine the vector potential.

Derivation of the Meissner effect

Start with the equation derived above (Eq.5) $\mathbf{j}_s = - e^2 n_s \mathbf{A}/m$ (5)

Use Maxwell equation MEq4: $\mathrm{curl}\mathbf{B} = \mu_0 \mathbf{j}_s$

Also, remember the definition of the vector-potential **A**. It is defined as curl**A**=**B**

Thus ,the new equation becomes:

$$(1/\mu_0)\ \mathrm{curl}\mathbf{B} = -(e^2 n_s /m)\ \mathbf{A}$$

$$(1/\mu_0)\ \mathrm{curl}(\mathrm{curl}\mathbf{B}) = -(e^2 n_s /m)\ \mathrm{curl}\mathbf{A} = -(e^2 n_s /m)\ \mathbf{B}$$

Remember that div**B**=0 (Meq.2). Therefore curl(curlB)=-Δ**B**

$$\nabla \times (\nabla \times A) \equiv curl\ curl\ A$$

$$\nabla\times(\nabla\times A) = \nabla(\nabla\cdot A) - \nabla^2 A$$

So, the equation for the magnetic field inside superconductor is:

$$\mathbf{(1/\mu_0)\ \Delta B = (\ e^2 n_s /m\ )\ B}$$

Let us take a simplified example. Suppose the magnetic field depends only on one coordinate, say on the *x*-axis. The direction of the magnetic field is along the *z*-axis. So, the magnetic field is $\mathbf{B}=B(x)\mathbf{e_z}$, where $\mathbf{e_z}$ is the unit vector along the *z*-axis. Then the equation becomes simpler:

$$(d^2/dx^2)B = (e^2 \mu_0 n_s /m)B$$

The solution is $\mathbf{B=B_0 exp(-x/\lambda_L)}$, where B_0 is the magnetic field at the surface and $(\lambda_L)^2 = m/(e^2 \mu_0 n_s)$ is the London penetration depth.

Let us make **a numerical estimate** for the penetration depth (in SI units):

$m=9.1\mathrm{x}10^{-31}$ kg is the electronic mass

$e=1.6\mathrm{x}10^{-19}$ C is the electronic charge

$\mu_0 = 4\pi \times 10^{-7}$ H/m is the magnetic constant

n_s is difficult to know exactly. We take the density of the conduction electrons in Nb: $n_s=5.56\mathrm{x}10^{28}$ m^{-3}

With this we get λ_L =23 nm. London length is a good estimate for the magnetic field penetration depth in the case of very pure, perfect crystal metallic samples. But it is not exact in this simple model.