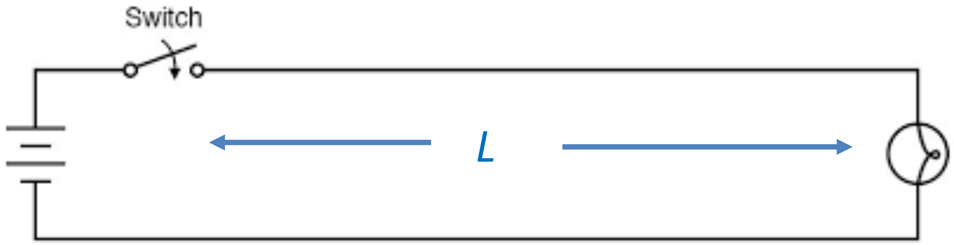
Transmission Lines

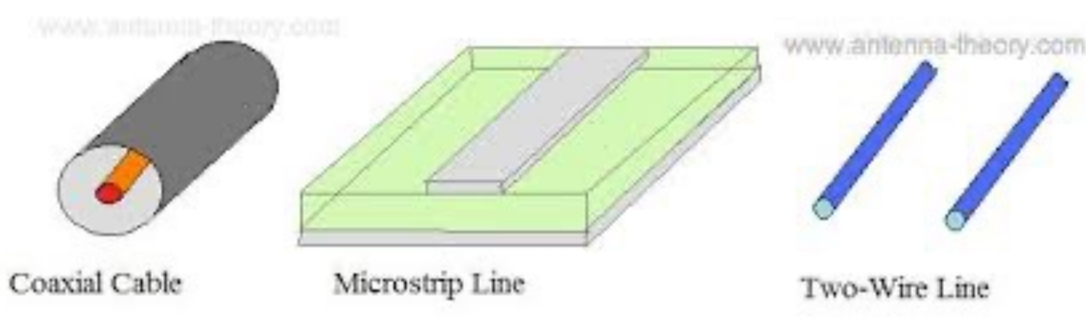
So far we have focused on *lumped* circuits, in which all the electromagnetic fields are confined (*lumped*) inside circuit elements such as resistors, capacitors, inductors, batteries and so on. The assumption is that the circuit components and the wires connecting them have negligible spatial extent. But obviously that can't always be true because the speed of light c = 3×10^8 m/sec is finite. In the circuit below, suppose L = 1 meter. When you close the switch it takes the light bulb $t = L/c \approx 3$ nsec to find out about it. If you were just turning on a light bulb you wouldn't notice a 3 nsec delay but if the switch and bulb were replaced by high-speed digital circuits then a 3 nsec delay would be easy to observe.



Or, look at it in the frequency domain. The assumption of lumped circuit analysis is that the size L of the circuit is much less than the free-space wavelength of light corresponding to the frequency at which the circuit operates,

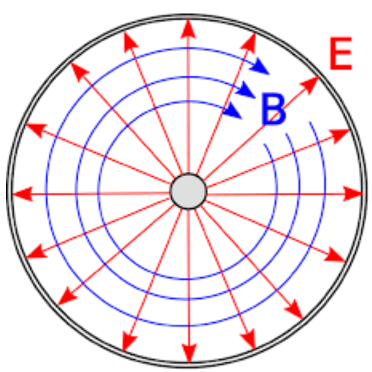
$$L \ll \lambda = \frac{c}{f}$$

If the battery were replaced by a sinewave generator running at f = 100 MHz (i.e., FM radio frequencies) then $\lambda = 3$ *m* and that is comparable to L. In that case, wave propagation along the connecting wires becomes important. In either case, we need to treat the wires connecting the generator to the load as a *distributed* circuit. That's the subject of transmission lines, the most common example being the ubiquitous coaxial cable.



We will focus on transmission lines like the ones shown on the left. All three consist of two metallic conductors separated by an insulator which could be vacuum or a dielectric. All three share one property – a constant cross-sectional shape in any plane perpendicular to the direction of propagation.

The coaxial cable is used in probably every physics lab in the world and is the easiest to analyze. Microstrip lines are used throughout printed circuitry. Two-wire lines were used back in the early days to connect the TV to the antenna for receiving network broadcasts. In each case an electromagnetic wave can propagate along the line. Its time-dependent electric and magnetic fields exist in the space localized around the conductors. The specific **E** and **B** field configurations will depend on the specific transmission line.



The figure shows the E and B fields in a cross-section of a coax cable. Imagine the coax inner conductor has radius a and uniform positive charge per unit length. The outer conductor has an inner radius b and carries an equal and opposite negative charge per unit length. Gauss's law leads to a radial electric field **E** and a voltage V between inner and outer conductor. Using Q = CV, It's easy to see that the coax will have a capacitance per unit length C_0 given by,

$$C_0(Farads/meter) = \frac{2\pi\epsilon_0}{ln\,\frac{b}{a}}$$

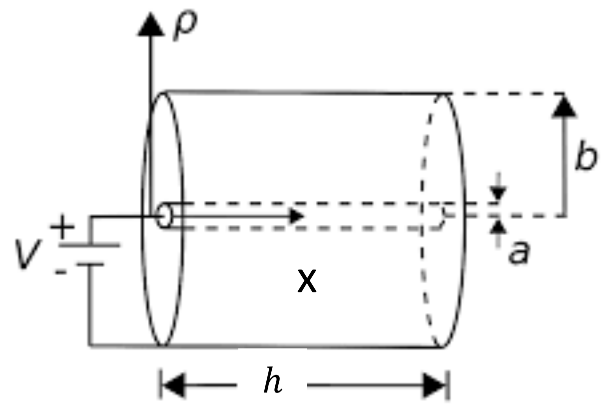
The coax will have an inductance per unit length L_0. To find that, imagine a current *I* coming out of the inner conductor and use Ampere's law to find **B**. To get the inductance per unit length use the energy stored in the **B** field for a length *h* of transmission line,

$$\frac{1}{2\mu_0}\int B^2\,dxdydz = \frac{1}{2}L_0 h\, I^2 \qquad \rightarrow \qquad L_0(Henries/meter) = \frac{\mu_0}{2\pi}\;ln\frac{b}{a}$$

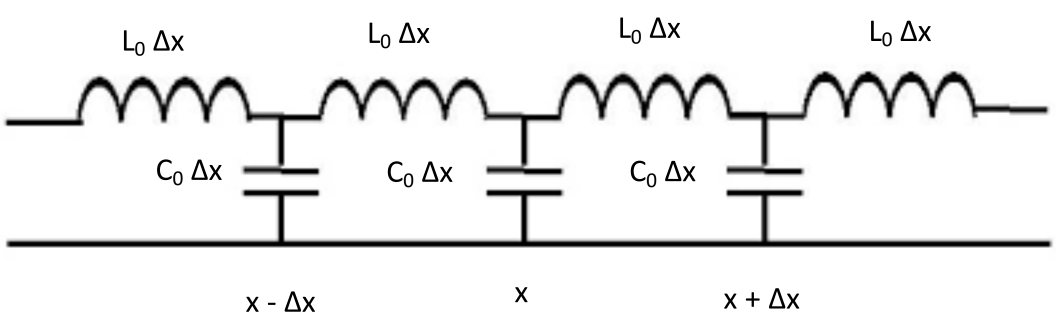
To find the inductance and capacitance per unit length for an arbitrarily-shaped transmission line you'll need to solve Maxwell's equations to get **E** and **B** and then relate them to the voltage and current via the energy equations:

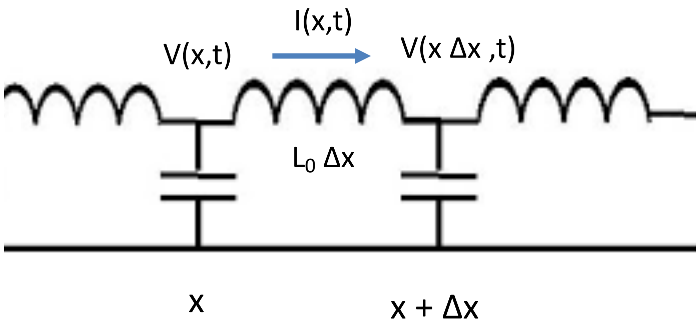
$$\frac{\varepsilon_o}{2}\int E^2\,dxdydz = \frac{1}{2}C_0 h\,V^2 \qquad \frac{1}{2\mu_0}\int B^2\,dxdydz = \frac{1}{2}L_0 h\,I^2$$

We will simply assume that for any transmission line there will be *some* C_0 and L_0 and go from there. As we will soon show, wavelike solutions for **E** and **B** will propagate back and forth in the x direction. Having given us C_0 and L_0, **E** and **B** will be put aside for the voltage V (x,t) between the two conductors and I (x,t) which is the current flowing through the inner conductor. Unlike the situation in lumped circuits, V and I now depend on *space* as well as time.



To understand the details of the propagation we'll treat the transmission line as a series of infinitely small capacitors and inductors. Each segment of the line with length Δx has a capacitance C_0 Δx and inductance L_0 Δx. For each of these infinitesimal *lumped* circuits we can apply Kirchoff's laws. This treatment can be found in many places, but I always recommend *The Feynman Lectures on Physics*, Vol. 2.





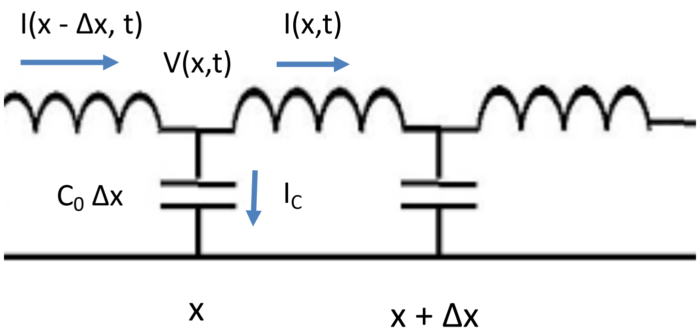
Focus on the little inductor between x and x + Δx. The voltage across it is given by,

$$V(x,t) - V(x+\Delta x,t) = L_0 \Delta x\,\frac{\partial I}{\partial t}$$

Expanding the left-hand side we have,

$$-\frac{\partial V}{\partial x}\Delta x = L_o \Delta x\,\frac{\partial I}{\partial t}$$

$$\boxed{-\frac{\partial V}{\partial x} = L_o\,\frac{\partial I}{\partial t}}$$



Next, focus on the capacitor $C_0 \Delta x$ located at x. Using current conservation, the current into the capacitor is,

$$I(x-\Delta x) - I(x) = I_C = C_0\,\Delta x\,\frac{\partial V}{\partial t}$$

Expanding the left side gives,

$$\boxed{-\frac{\partial I}{\partial x} = C_0\,\frac{\partial V}{\partial t}}$$

These two boxed equations are known as the the *Telegrapher's Equations*. They are essentially Faraday's Law and Ampere's Law in the context of transmission lines. Taking the space derivative of the first equation, the time derivative of the second and setting the mixed partial derivatives of *I* to be equal, we get the wave equation:

$$\boxed{\frac{\partial^2 V}{\partial x^2} = L_0 C_0\,\frac{\partial^2 V}{\partial t^2}}$$

There is an identical wave equation for the current *I*(x,t). The transmission line supports wave-like solutions of *V* and *I* in which the phase velocity of the wave is given by,

$$\tilde{c} = \frac{1}{\sqrt{L_0 C_0}}$$

It can be shown that if the space surrounding the conductors of the transmission line is free of dielectrics, then $\tilde{c} = c$, the speed of light in a vacuum. Usually there is a dielectric around, in which case C_0 is proportional to the dielectric constant ϵ and the velocity is reduced by $1/\sqrt{\epsilon}$. For coax cables, which typically have a Teflon-like dielectric between the inner and outer conductor, $\tilde{c} \approx 0.6\, c$.

Characteristic Impedance

Since V and I both obey the wave equation, which is linear, we can use phasor analysis to examine waves at a particular angular frequency. Represent the physical voltage and current along the line by the real part of phasors,

$$\mathrm{V}(x,t) = Re\left(\hat{V} e^{i(\omega t - kx)}\right) \qquad \mathrm{I}(x,t) = Re\left(\hat{I} e^{i(\omega t - kx)}\right)$$

To find *k,* substitute $e^{i(\omega t - kx)}$ into the wave equation, take the derivatives and cancel $e^{i(\omega t - kx)}$ from both sides:

$$\frac{\partial^2}{\partial x^2} e^{i(\omega t - kx)} = \frac{1}{\tilde{c}^2} \frac{\partial^2}{\partial t^2} e^{i(\omega t - kx)} \qquad \omega^2 = \tilde{c}^2 k^2 \quad \rightarrow \quad k = \pm \frac{\omega}{\tilde{c}}$$

For a given frequency ω (assumed to be positive) there are waves travelling to the right $(k = \omega/\tilde{c})$ and waves travelling to the left $(k = -\,\omega/\tilde{c})$. The complete solution on a transmission line generally involves 4 waves: right and left going voltage and right and left going current waves. But things simplify if we take a right-going voltage and current wave (denoted by a (+) subscript) and plug this solution into either one of the Telegrapher equations:

$$-\frac{\partial}{\partial x} \hat{V}_+ e^{i(\omega t - kx)} = L_o \frac{\partial}{\partial t} \hat{I}_+ e^{i(\omega t - kx)} \quad \rightarrow \quad k\, \hat{V}_+ = \omega L_0\, \hat{I}_+$$

Using the previous expression for the phase velocity, the ratio of the complex voltage amplitude to the complex current amplitude is given by,

$$\frac{\hat{V}_+}{\hat{I}_+} = \frac{\omega L_0}{k} = \frac{\omega L_0}{\omega/\tilde{c}} = L_0 \tilde{c} = L_0 \frac{1}{\sqrt{L_0 C_0}} = \sqrt{\frac{L_0}{C_0}} = Z_0$$

This ratio of complex amplitudes has the dimensions of Ohms and is called the *characteristic impedance*. If we now go through the calculation for a left-going wave we find that the characteristic impedance is $-Z_0$. Denoting the right and left going complex amplitudes with a (+) or (–) subscript we have,

$$\frac{\hat{V}_+}{\hat{I}_+} = Z_0 \qquad \frac{\hat{V}_-}{\hat{I}_-} = -Z_0$$

Knowing the complex voltage amplitudes for right and left going voltage waves, we automatically know the corresponding amplitudes of the current waves. The general solution at a given frequency is a sum of right and left-going waves,

$$V(x,t) = Re\left(\hat{V}_+ e^{i(\omega t - kx)} + \hat{V}_- e^{i(\omega t + kx)}\right) \qquad I(x,t) = Re\left(\hat{I}_+ e^{i(\omega t - kx)} + \hat{I}_- e^{i(\omega t + kx)}\right)$$

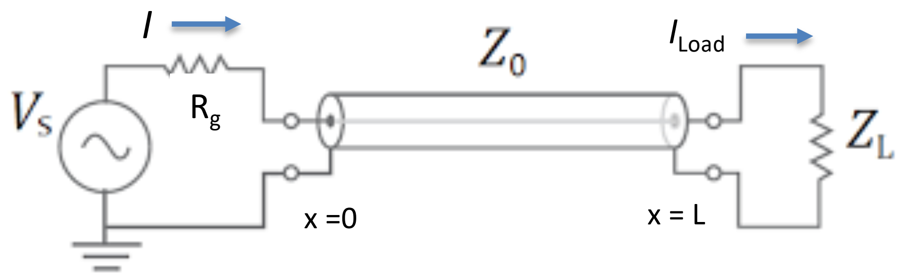
Using the characteristic impedance expressions, everything can be put in terms of just the right and left going voltage amplitudes,

$$V(x,t) = Re\left(\hat{V}_+ e^{i(\omega t - kx)} + \hat{V}_- e^{i(\omega t + kx)}\right) \qquad I(x,t) = Re\left(\frac{\hat{V}_+}{Z_0} e^{i(\omega t - kx)} - \frac{\hat{V}_-}{Z_0} e^{i(\omega t + kx)}\right)$$

To obtain the (+) and (-) voltage amplitudes will require two boundary conditions. Although Z_0 has the dimensions of Ohms, it is *not* a real resistance that dissipates energy! Our transmission line model contained no resistors (although a more realistic model would include them). Z_0 depends on the capacitance and inductance per unit length so it depends on the specific cross-sectional shape of the transmission line. Coax cables typically have $Z_0 \approx 50\ \Omega$. Two-wire lines such as old-fashioned antenna cable might have $Z_0 \approx 300\ \Omega$. Characteristic impedances typically vary from about 10 – 500 Ohms, depending on the transmission line geometry. The variation is not large because Z_0 usually varies logarithmically with the cable dimensions. For example, in coax cables Z_0 varies as $\ln(R_2/R_1)$ where R_1 and R_2 correspond to the radii of the inner and outer conductor, respectively. For our transmission line model Z_0 is *real* (i.e., resistive). If we were to include some actual resistance, along with L_0 and C_0, in the transmission line model, then Z_0 would acquire an imaginary part. However, for a great deal of high frequency electronics, treating Z_0 as real is accurate enough.

Boundary values

The next problem is to find $\hat{V}_+$ and $\hat{V}_-$. Focus on the circuit shown below. Without the transmission line this is just a voltage divider. The transmission line is a distributed circuit that connects the lumped circuits at x = 0 and x = *L*. The assumption in such schematics is that the lumped circuits on either end have no spatial extent and can be treated with Kirchoff's voltage and current laws. The transmission line piece has the general solution we just wrote down, with two unknown amplitudes. The circuits on either end provide the two boundary conditions.



Focus first on the situation at x = 0. The current from the generator must equal the total transmission line current at x = 0. For the transmission line current, set x = 0 in each exponential. As usual, all the $e^{i\omega t}$ factors cancel out and we have,

$$\hat{I} = \hat{I}_+ + \hat{I}_- = \frac{\hat{V}_+}{Z_0} - \frac{\hat{V}_-}{Z_0}$$

Using Kirchoff's voltage law at x = 0 we also have $\hat{V}_S - \hat{I}R_g = \hat{V}_+ + \hat{V}_-$. Before solving set $R_g = Z_0$. In other words, the generator output impedance is *purposely* made equal to the characteristic impedance of the transmission line, typically 50 Ohms. That's typical of most high frequency electronics. With that,

$$\hat{V}_+ = \frac{\hat{V}_S}{2}$$

This result has a simple physical interpretation. If $Z_0 = R_g$, then the transmission line divides the generator voltage by ½. If, for example, $R_g = Z_0$ = 50 Ohms, then the transmission line looks like a 50 Ohm resistor to the generator. However, this holds true *only* so long as there is only a right-going wave present at x = 0. That can happen if the generator produces a voltage step at t = 0. The step voltage travels down the line, reflects off the far end and travels back in a total time $T = 2L/\tilde{c}$. Until that reflected wave reaches x = 0, the generator doesn't know about it and the transmission line looks like a resistor of value Z_0. But after that, we need to add the reflected wave.

We still need to solve for the left-going voltage amplitude. To do that, go to x = L. Again, the total current on the line at x = L must equal the current through the load impedance. Using Kirchoff's current law,

$$\hat{I}_+ e^{-ikL} + \hat{I}_- e^{ikL} = \hat{I}_{Load}$$

The total voltage on the line at x = L must equal the voltage across the load:

$$\hat{V}_+ e^{-ikL} + \hat{V}_- e^{ikL} = Z_L \hat{I}_{Load}$$

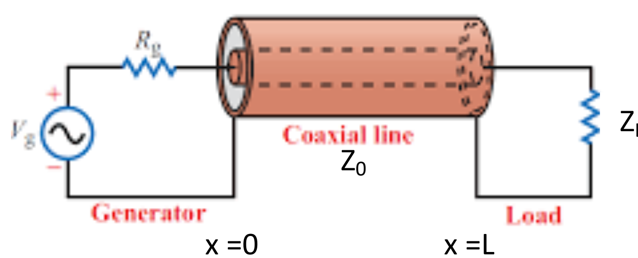
Solving for the ratio of the complex voltage amplitudes we find,

$$\frac{\hat{V}_-}{\hat{V}_+} = e^{-2ikL}\frac{Z_L - Z_0}{Z_L + Z_0} = e^{-2ikL}\,\Gamma_V$$

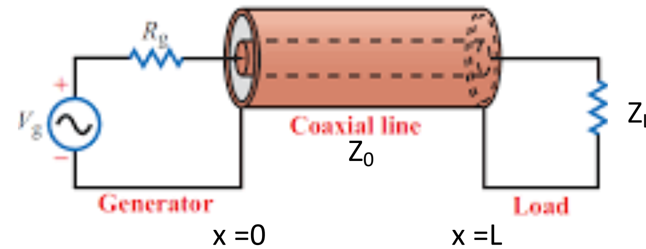
Γ_V is known as the *voltage reflection coefficient*. In general, Γ_V is complex but there is one important case where it is zero. That occurs when $Z_L = Z_0$, the characteristic impedance of the line in which case there is *no reflected wave*. This is called *terminating* the transmission line in its characteristic impedance. It's done everywhere in high frequency circuits to avoid reflections which cause standing waves and create problems. The transmission lines often have $Z_0 = 50\ \Omega$ so both the input and output impedance of amplifiers (and other components) are purposely made to be $50\ \Omega$. If such an amplifier is connected to the line at x = L there will be no reflected wave.

We now have everything to determine the currents and voltages in the circuit. The full solution is given by,

$$V(x,t) = Re\left(\hat{V}_+ e^{i(\omega t - kx)} + \hat{V}_- e^{i(\omega t + kx)}\right)$$
$$I(x,t) = Re\left(\frac{\hat{V}_+}{Z_0} e^{i(\omega t - kx)} - \frac{\hat{V}_-}{Z_0} e^{i(\omega t + kx)}\right)$$



Assume the generator is built with $R_g = Z_0$. Also use the expression for the ratio of the left to right going voltage waves. The voltage is given by,



$$V(x,t) = Re\left(\hat{V}_+ e^{i(\omega t - kx)} + \hat{V}_- e^{i(\omega t + kx)}\right) = Re\left(\frac{\hat{V}_S}{2}\, e^{i(\omega t - kx)} + \frac{\hat{V}_S}{2}\, \Gamma_V\, e^{-2ikL} e^{i(\omega t + kx)}\right)$$

$$V(x,t) = Re\left(\frac{\hat{V}_S}{2}\, e^{i\omega t}\left[e^{-ikx} + \Gamma_V e^{-2ikL} e^{ikx}\right]\right)$$

Similarly, the current is given by,

$$I(x,t) = Re\left(\frac{\hat{V}_+}{Z_0} e^{i(\omega t - kx)} - \frac{\hat{V}_-}{Z_0} e^{i(\omega t + kx)}\right) = Re\left(\frac{\hat{V}_S}{2\, Z_0}\, e^{i\omega t}\left[e^{-ikx} - \Gamma_V e^{-2ikL} e^{ikx}\right]\right)$$

We usually aren't probing along the transmission line to measure V and I at each x but instead, we care about the behavior at x = 0 and x = L where things are connected. For example, find the impedance looking into the line at x = 0. That's the ratio of the full complex voltage amplitude divided by the full complex current amplitude at x = 0:

$$Z(x=0) = \frac{\hat{V}(x=0)}{\hat{I}(x=0)} = \frac{\frac{\hat{V}_S}{2}\left[e^{-ikx} + \Gamma_V e^{-2ikL} e^{ikx}\right]}{\frac{\hat{V}_S}{2\, Z_0}\left[e^{-ikx} - \Gamma_V e^{-2ikL} e^{ikx}\right]} = Z_0\, \frac{1 + \Gamma_V e^{-2ikL}}{1 - \Gamma_V e^{-2ikL}} = Z_0\, \frac{Z_L + i\, Z_o \tan kL}{Z_0 + i\, Z_L \tan kL}$$

This last formula is particularly useful and we will consider three different cases.

1. Terminated Line: First consider the case $Z_L = Z_0$ in which case $\Gamma_V = 0$ and there will be no reflected wave. For any length of line, we have,

$$Z(x=0) = Z_0 \qquad\qquad Z_L = Z_0$$

As stated earlier, when there is no reflected wave, then no matter what the length of the transmission it appears to the generator like an impedance Z_0 .

2. Half-wave line: Suppose the line is one half-wavelength long, $L = \lambda/2$. Then $k = 2\pi/\lambda$ so $k = 2\pi/2L$ and,

$$\tan kL = \tan \pi = 0 \quad \rightarrow \quad Z(x=0) = Z_L$$

In other words, if the line has length $L = \lambda/2$ (or any integral multiple of that) then the generator thinks the load impedance is connected right at x = 0. Of course in the *time* domain, the signal still takes a finite time to reach the load. But in the steady state situation with a signal generator operating at constant frequency, the half-wave line is invisible.

3. Quarter-wave line: Now consider a line with length $L = \lambda/4$. Then $k = \pi/2L$ and we have,

$$\tan kL = \tan\frac{\pi}{2} = \infty \quad \rightarrow \quad Z(x=0) = \frac{Z_0^2}{Z_L}$$

The transmission line inverts the load impedance. If a quarter-wave line is short-circuited at x = L then $Z_L = 0$ and,

$$Z(x=0) = \infty$$

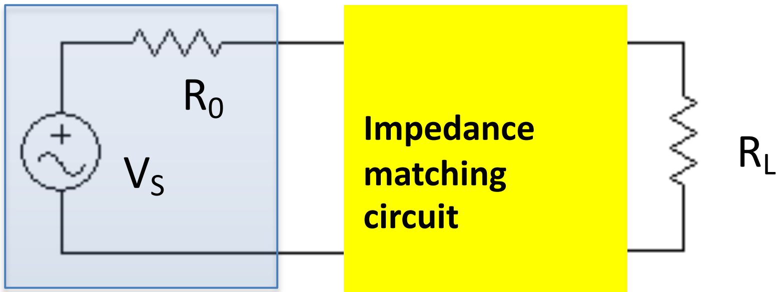
To the generator, the line looks like an *open* circuit! Similarly, if a quarter wave line is open-circuited at x = L then the generator thinks the line is short circuited. The quarter-wave line acts like an *impedance transformer*. If the load is a capacitor then,

$$Z_L = \frac{1}{i\omega C} \;\rightarrow Z(x=0) = Z_0^2 i\omega C = i\omega L \quad L = Z_0^2 C$$

The line has effectively transformed a capacitor into an inductor. Or, it will transform an inductor into a capacitor.

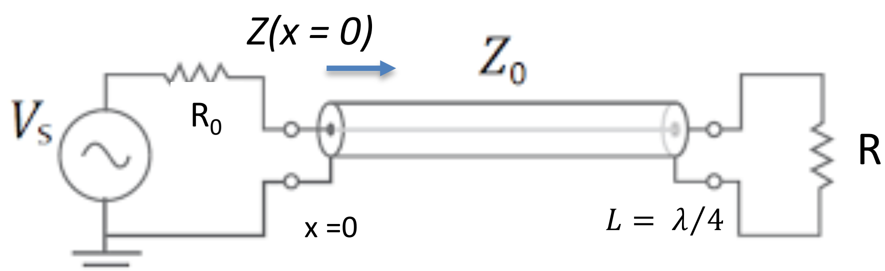
Impedance matching

Suppose we have a load resistance R_L to which we wish to transfer the maximum *power*. If the generator has the effective circuit shown in the shaded box, then the maximum power it can deliver to a load occurs when $R_0 = R_L$. However, R_L is generally *not* equal to R_0 so what can be done? The solution is an impedance-matching circuit that tricks the generator into thinking that it's connected to a load of resistance R_0. You can use a transmission line to do it.



Once again, exploit the impedance transformation properties of a $L = \lambda/4$ transmission line:

$$\tan kL = \tan\frac{\pi}{2} = \infty \quad \rightarrow \quad Z(x=0) = \frac{Z_0^2}{R_L}$$



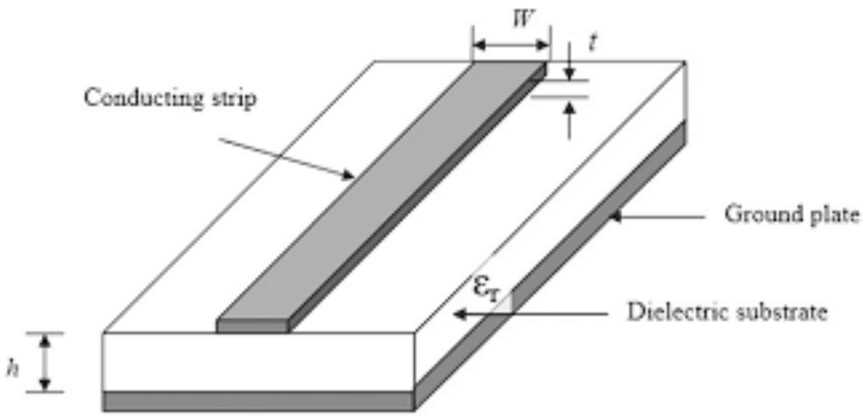
To impedance match R_L to the generator we choose,

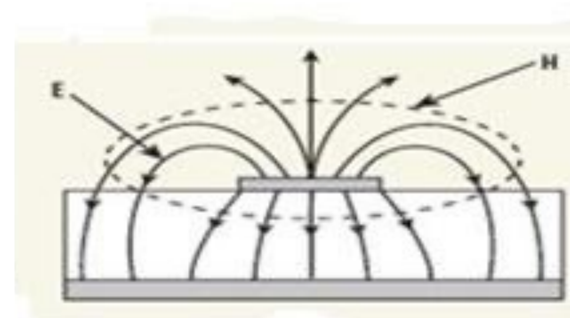
$$Z(x=0) = \frac{Z_0^2}{R_L} = R_0 \rightarrow Z_0 = \sqrt{R_L R_0}$$

To impedance match the generator to the load, you need to make a transmission line with Z_0 equal to the geometric mean of R_0 and R_L. And it needs to be a quarter wavelength long at the frequency where you wish to operate. But how to make a transmission line with a specific characteristic impedance? The next slide shows one way.

Microstrip

Microstrip is a transmission line configuration that is widely used in high frequency circuits. It consists of a conducting strip sitting on a dielectric, beneath which is a conducting ground plane. The right-hand figure shows the configuration of **E** and **H** fields.





The nice thing about microstrip is that by adjusting the ratio *W*/*h* you can set the characteristic impedance Z_0. An approximate formula is,

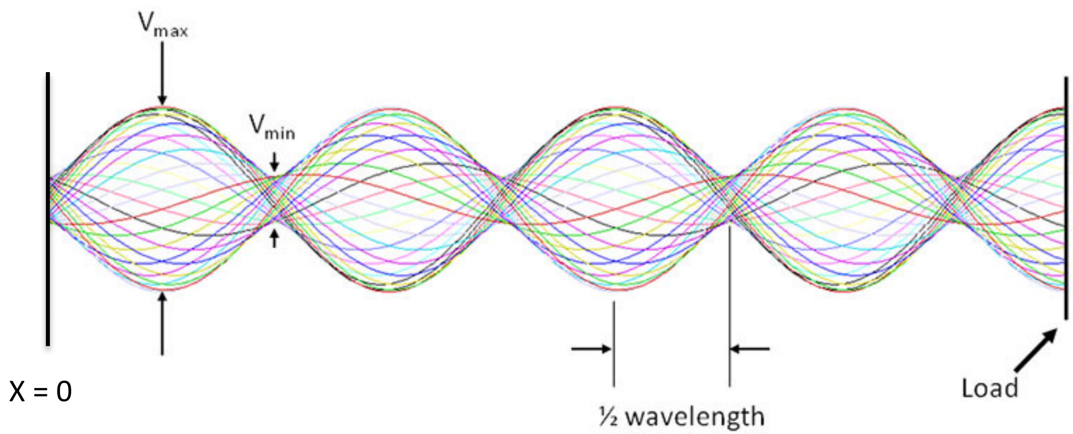
$$Z_0(Ohms) \approx \frac{377}{\left(\frac{W}{h}+1\right)\sqrt{\epsilon_r+\sqrt{\varepsilon_r}}}$$

As the frequency moves toward 1 GHz and beyond, this approximate formula no longer holds for several reasons:

(1) the dielectric constant is frequency-dependent and develops a resistive (i.e., loss) component.
(2) The transmission line model must include dielectric loss and a frequency-dependent resistance due to the skin depth of the metal.
(3) The field configuration shown is a transverse electromagnetic (TEM) wave in which E and H are perpendicular to the direction of propagation. This configuration holds true so long as W and h are much less than the wavelength of the wave. This, in turn, is typically valid for frequencies well below 10-15 GHz, after which non-TEM modes may appear and complicate things considerably.

Standing waves

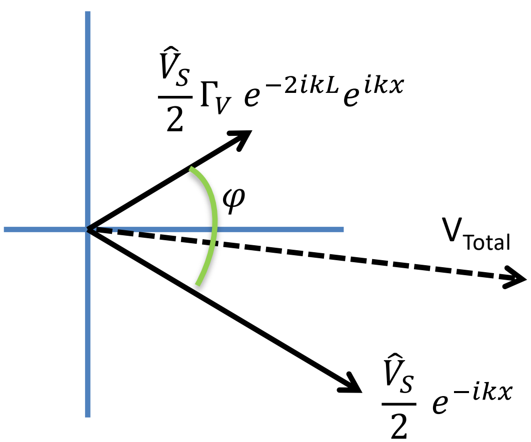
If right-going and left-going waves are both present on the line there will be standing waves. Assuming we have a sinusoidal generator voltage, the voltage on the line V(x,t) would look something like this figure where different colors indicate different times. At every point the voltage oscillates at ω and the amplitude of the oscillation varies sinusoidally with x. It's a standing wave.



From: nutsvolts.com

To derive the envelope function of the standing wave, use the solution for the voltage on the line:

$$V(x,t) = Re\left(\frac{\hat{V}_S}{2}\, e^{i\omega t}\left[e^{-ikx} + \Gamma_V e^{-2ikL} e^{ikx}\right]\right)$$



The expression inside the parentheses is the sum of two phasors, one of length $V_S/2$ (proportional to the right-going wave) and one of length $|\Gamma_V|V_S/2$ (proportional to the left-going wave). This total phasor, represented by the dashed vector, rotates counterclockwise at ω. Its projection on the x-axis is the physical voltage on the line. To find the sum, write the reflection coefficient as $\Gamma_V = |\Gamma_V|e^{i\theta}$. The phase difference between the two solid phasors is,

$$\varphi = kx + \theta - 2kL - (-kx) = 2k(x-L) + \theta$$

Now use the law of cosines to find the length of V_{total} :

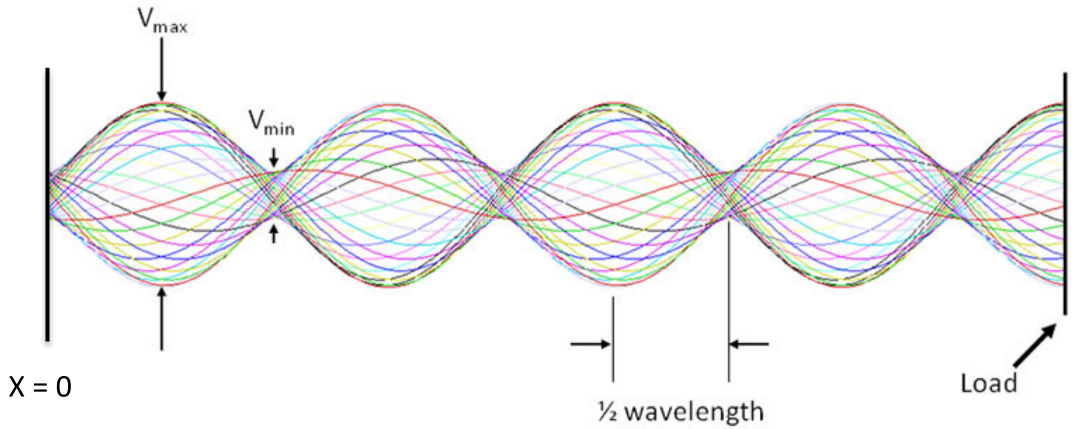
$$|V_{total}|^2 = \left|\frac{V_S}{2}\right|^2 + \left|\frac{V_S}{2}\Gamma_V\right|^2 + 2\left|\frac{V_S}{2}\right|^2 |\Gamma_V|\, cos\varphi = \left|\frac{V_S}{2}\right|^2 \left(1 + |\Gamma_V|^2 + 2|\Gamma_V|\, cos(2kx - 2kL + \theta)\right)$$

This quantity varies as we move along x because the cosine changes from -1 to + 1.

$$|V_{Total}|_{Max} = \left|\frac{V_S}{2}\right| (1 + 2|\Gamma_V| + |\Gamma_V|^2)^{\frac{1}{2}} = \left|\frac{V_S}{2}\right| (1 + |\Gamma_V|)$$

$$|V_{Total}|_{Min} = \left|\frac{V_S}{2}\right| (1 - 2|\Gamma_V| + |\Gamma_V|^2)^{\frac{1}{2}} = \left|\frac{V_S}{2}\right| (1 - |\Gamma_V|)$$

The standing waves have a minimum and maximum amplitude as shown below,



From: nutsvolts.com

The ratio of these two values is known as the voltage standing wave ratio, known as VSWR:

$$VSWR = \frac{V_{Max}}{V_{Min}} = \frac{(1 + |\Gamma_V|)}{(1 - |\Gamma_V|)}$$