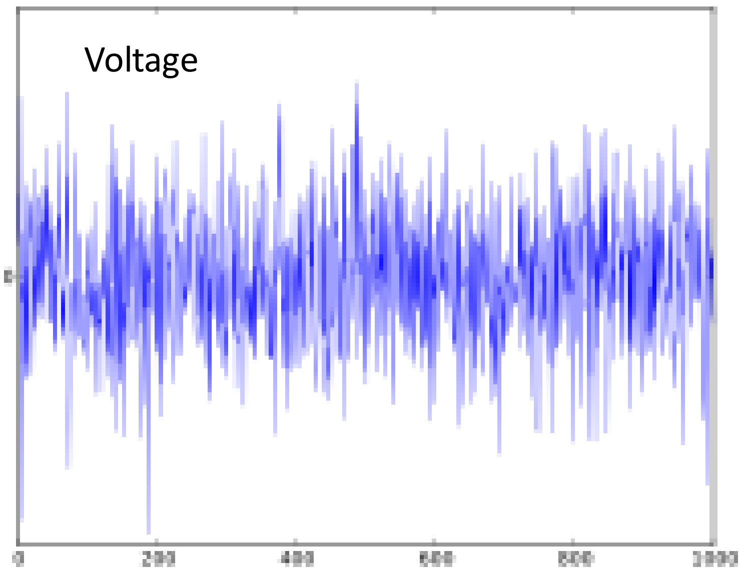
Electronic Noise



If you were to look at the voltage versus time across any resistor you would see something similar to the trace on the left. This *noise* voltage is a fluctuating quantity whose time average over a sufficiently long time interval t_{av} is zero.

$$\bar{V} = \frac{1}{t_{av}} \int_{t_0}^{t_0+t_{av}} V(t)\, dt = 0$$

However, the average of V^2 (the *mean squared* value) is *not* zero:

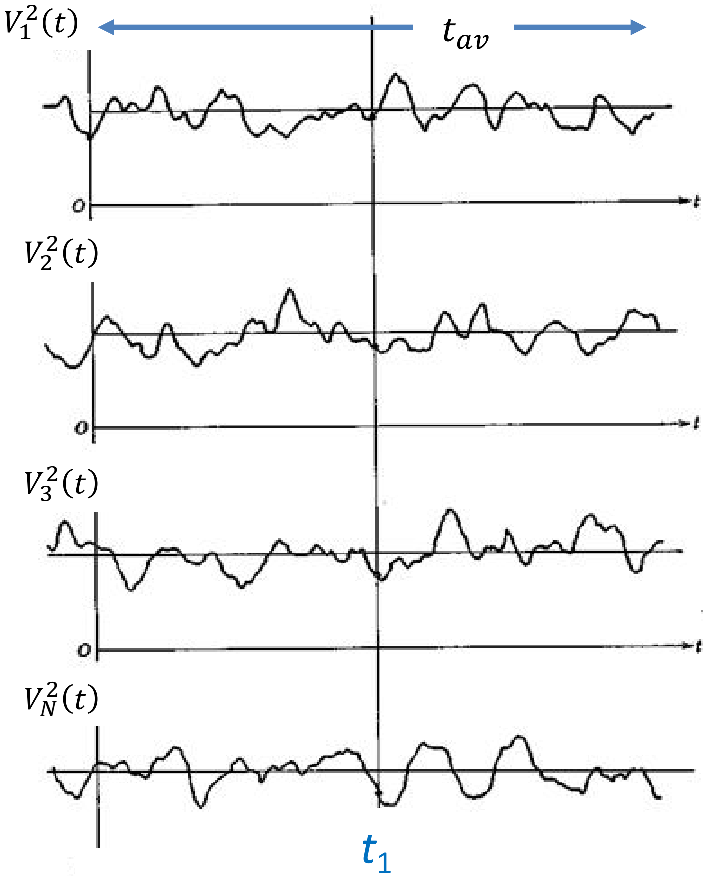
$$\overline{V^2} = \frac{1}{t_{av}} \int_{t_0}^{t_0+t_{av}} V^2(t)\, dt \neq 0$$

For the vast majority of cases in electronics, it doesn't matter when the time averaging begins so t_0 is arbitrary. When that's true, the noise is called *stationary*. We'll assume that's true from here on. Assume for now that t_{av} is much longer than any other characteristic time in the system.

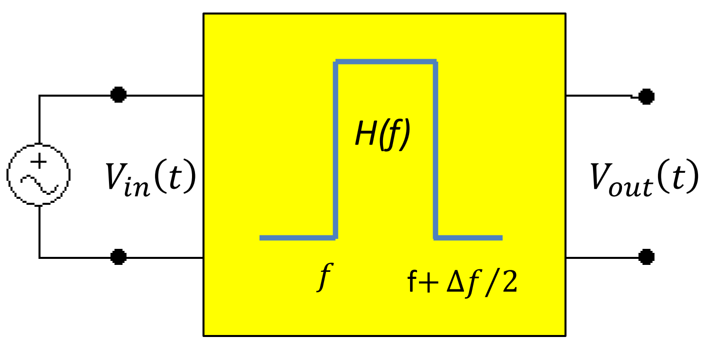
There are different kinds of averaging for a physical system. For example, from statistical mechanics we know that the average squared velocity in an ideal gas is given by,

$$\langle v^2 \rangle = \int_0^\infty v^2 P(v)\, dv = \frac{3\, k_B T}{m}$$

This is an *ensemble* average, denoted by the brackets. In this case the average is over a huge number of otherwise identical systems. From that analysis we obtain a Maxwell-Boltzmann probability distribution $P(v)$ for the speed. It's not obvious that time and ensemble averages will always give the same answer and it depends on how long we do the time average for. In an ideal gas, we'd need t_{av} much larger than a typical collision time between molecules. When the two averages *do* give the same answer the system is termed *ergodic*. There is a large literature on this subject but we will assume it's true and make use of both averages.



http://heli-air.net/2016/02/02/ensemble-average/



To illustrate the two averages in an electronics context, imagine measuring the mean squared noise voltage at the output of a circuit. The time average might be taken on the top trace with an averaging time t_{av}.

$$\overline{V^2} = \frac{1}{t_{av}} \int_0^{t_{av}} V_1^2(t)\, dt$$

For the ensemble average, envision *N* identical circuits where *N* is huge. Now take a vertical cut any time t_1. The ensemble average is,

$$\langle V^2 \rangle = \frac{1}{N} \sum_{i=0}^{N} V_n^2(t_1) \qquad N \to \infty$$

Again, if the system is stationary it doesn't make any difference what we choose for t_1 so $\langle V^2 \rangle$ is time independent.

Since the frequency domain makes the analysis of circuits so easy, we would like to use it to analyze noisy electronics. For this we need a new quantity. Imagine a s noise generator $V_{in}(t)$ connected to a filter with a transfer function *H*(*f*) . For this filter, *H* = 1 if the frequency is between f and $f + \Delta f/2$, and *H* = 0 otherwise. If $\Delta f \ll f$ then define the *power spectral density* $S_{in}(f)$ as,

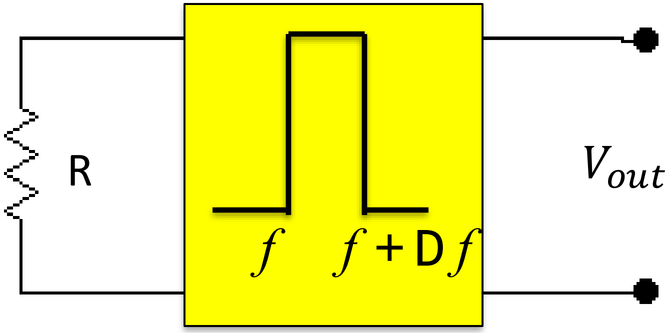
$$\langle V_{out}^2 \rangle = S_{in}(f)\, \Delta f$$

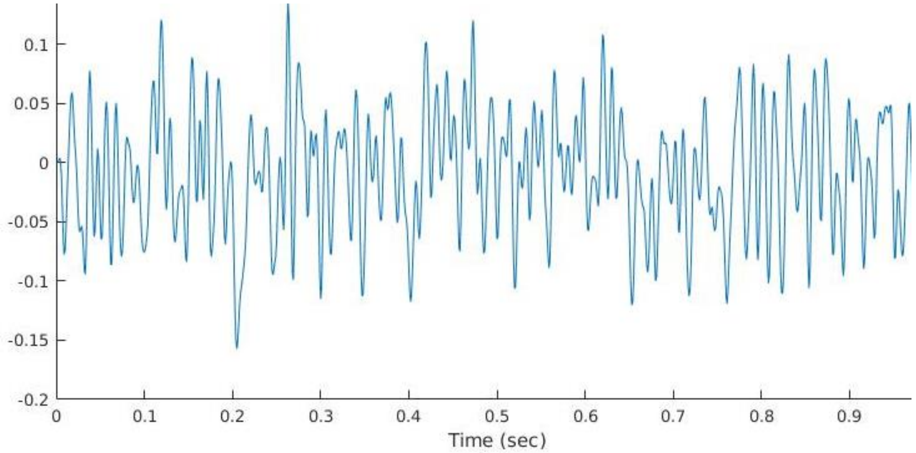
The mean-squared output noise depends on the bandwidth of the filter. For an arbitrary filter, we'll show that the output noise is,

$$\langle V_{out}^2 \rangle = \int_0^{\infty} |H(f)|^2\, S_{in}(f)\, df$$

Nyquist Noise

The noise voltage that appears across a resistor in thermal equilibrium is named after Harry Nyquist and John B. Johnson of Bell Telephone Labs. Johnson identified it experimentally and Nyquist used statistical mechanics to explain its origin. (See, for example, wikipedia.org/wiki/Johnson%E2%80%93Nyquist_noise.) It is the electronic analog of black-body radiation.





The basic idea is shown above. A resistor in equilibrium at absolute temperature T is connected to a filter that rejects all frequencies but those in a narrow band between f and $f + \Delta f$. Nyquist proved that the mean-squared noise on the output of this filter is given by,

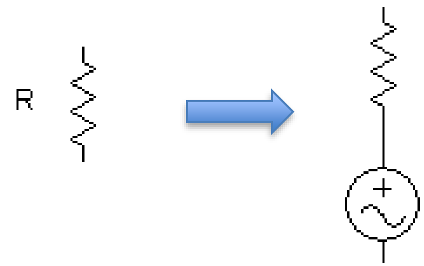
$$\langle V_{out}^2 \rangle = S_{Nyquist}(f)\,\Delta f = 4k_B T R\,\Delta f$$

As it stands, the spectral density of Nyquist noise is *independent* of frequency. Noise of that variety is known as *white* noise. In reality, there will be quantum corrections to the spectral density but they only appear at frequencies of order $f = k_B T/h$ where h is Planck's constant. At T = 300 K that corresponds to frequencies above 10^{12} Hz, which is much too high for conventional electronics. For practical purposes, Nyquist noise has a constant spectral density.

Nyquist noise is small but certainly observable. Consider the noise across an R = 1 MΩ resistor at T = 300 K viewed with a filter of bandwidth Δf = 10^8 Hz, which is about the bandwidth of many oscilloscopes,

$$\langle V_{out}^2 \rangle = S_{Nyquist}(f)\,\Delta f = 4k_B TR\,\Delta f = 1.66\ x\ 10^{-6}\ Volts^2$$

$$V_{RMS} = \sqrt{\langle V_{out}^2 \rangle} = 1.29\ x\ 10^{-3}\ Volts$$

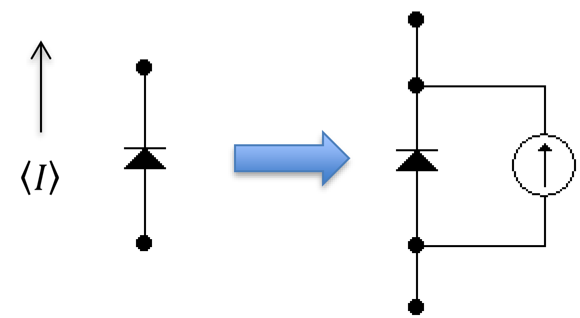


$$\langle V_{out}^2 \rangle = 4k_B TR\,\Delta f$$

Our ultimate goal is to estimate the noise in electronic circuits and compare it to the signal. For noise calculations, any resistor R acts like a noiseless resistor R in series with a random voltage generator whose mean squared value is just the Nyquist expression. We can't predict the instantaneous voltages and currents in such a circuit but we can predict the mean squared values.

Shot Noise

Shot noise is a fluctuation in the *current* arising from the discreteness of charge. It typically arises when the the charge must surmount an energy barrier as in a diode or transistor. For reasons somewhat beyond the scope of this discussion, it does not appear in resistors. In its simplest realization, shot noise is also *white*, with a spectral density $S_{shot} = 2q\langle I \rangle$,



$$\langle I_{shot}^2 \rangle = \langle (I - \langle I \rangle)^2 \rangle = S_{shot}\,\Delta f = 2q\langle I \rangle \Delta f$$

Here $\langle I \rangle$ is the average current flowing, as determined by non-random (i.e., deterministic) voltage and current sources in the circuit. q is the charge carrying the current. Normally this would be just the charge e of an electron but in some exotic condensed matter systems (i.e., fractional quantum Hall effect) q might be e/3. Shot noise typically acts like a noisy current generator in parallel with the average current flow. It's important in things like photodiodes, as shown in the figure.

Shot noise in ordinary circuits is rather small. Consider an average DC current < I > = 1 amp. The charge q = e = 1.6 x 10^{-19} C. Suppose we measure the current with an instrument of bandwidth Δf = 10^2 Hz. Then the root-mean-squared shot noise current is,

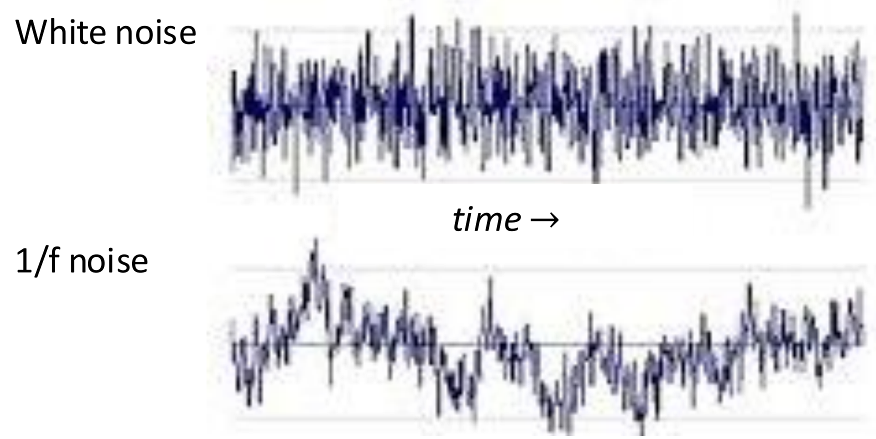
$$\sqrt{\langle I_{shot}^2 \rangle} = \sqrt{2e\,\langle I \rangle\,\Delta f} = 5.7\,x10^{-9}\,Amps$$

Interestingly, you can measure the electronic charge e using shot noise. Shot noise has been compared to what you hear under an umbrella in the rain. There is an average rainfall hitting the umbrella, analogous to $\langle I \rangle$. The pinging sounds correspond to individual raindrops, which arrive in discrete amounts.

1/f Noise

Not all noise is white. *Flicker* noise, sometimes called *pink* noise, is seen in things ranging from the incidence of musical notes in a symphony to fluctuations in the Nile river height. In semiconductor circuits, flicker noise voltage has a spectral density,

$$S_{1/f} \sim \frac{1}{f^{\alpha}} \quad 1 < \alpha < 2$$



When $\alpha \approx 1$ it's called *1/f* noise. There is no universal explanation for it but charge jumping in and out of energy traps in a semiconductor will give rise to noise fluctuations with a similar spectrum. The figure shows the difference between white noise and 1/*f* noise in the time domain.

1/f noise is difficult to reduce. As you probably would guess, white noise can be reduced by averaging. The longer you average the lower the noise. That's equivalent to reducing the bandwidth. But if the noise has a 1/f spectrum then the longer your averaging time the *larger* the fluctuations. It's therefore advisable, if possible, to work at frequencies well away from zero.

The figure below shows the contributions of white noise and flicker noise in a typical amplifier. Above some corner frequency, in this case about 1 kHz, white noise dominates. If possible, it's best to operate at frequencies above this value. We'll see how to do that with a lockin amplifier.

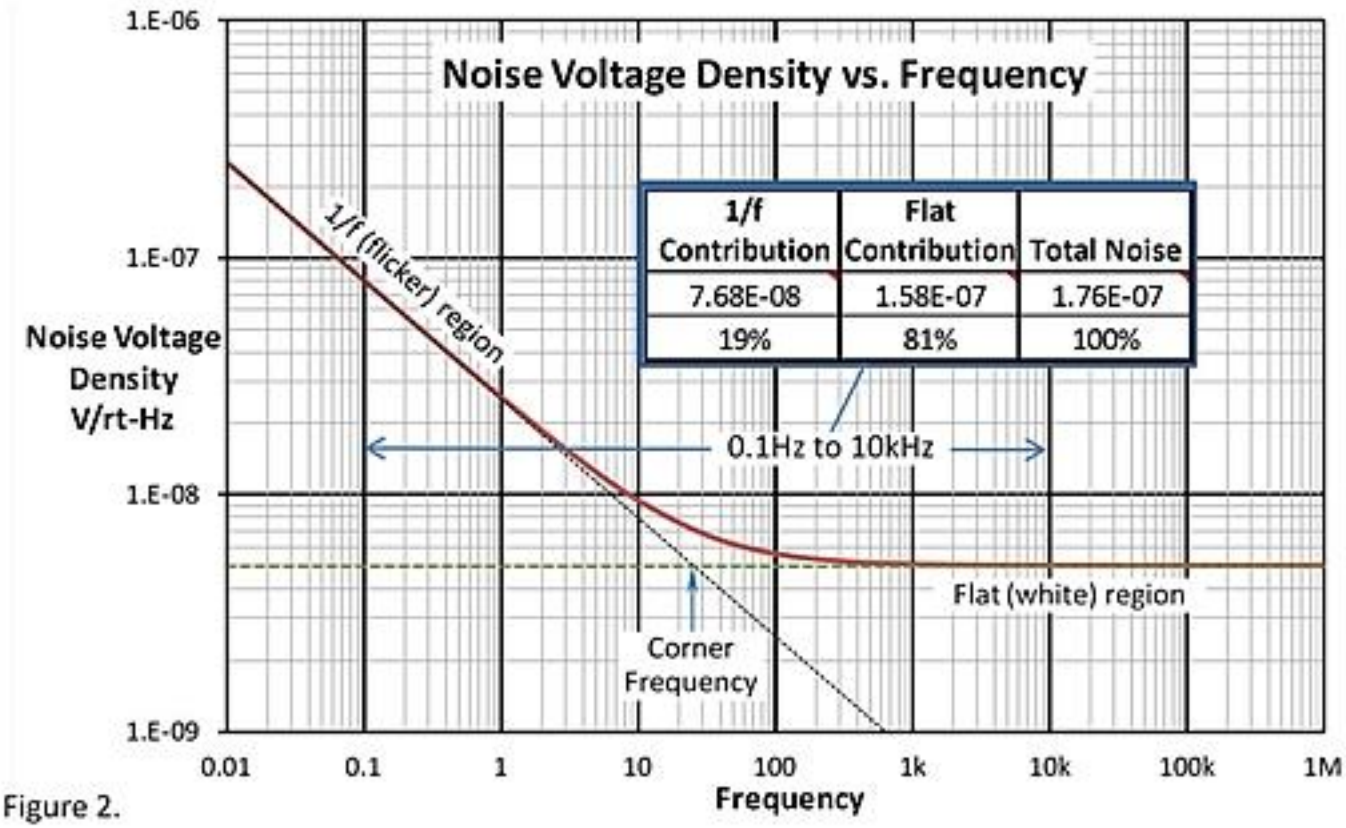


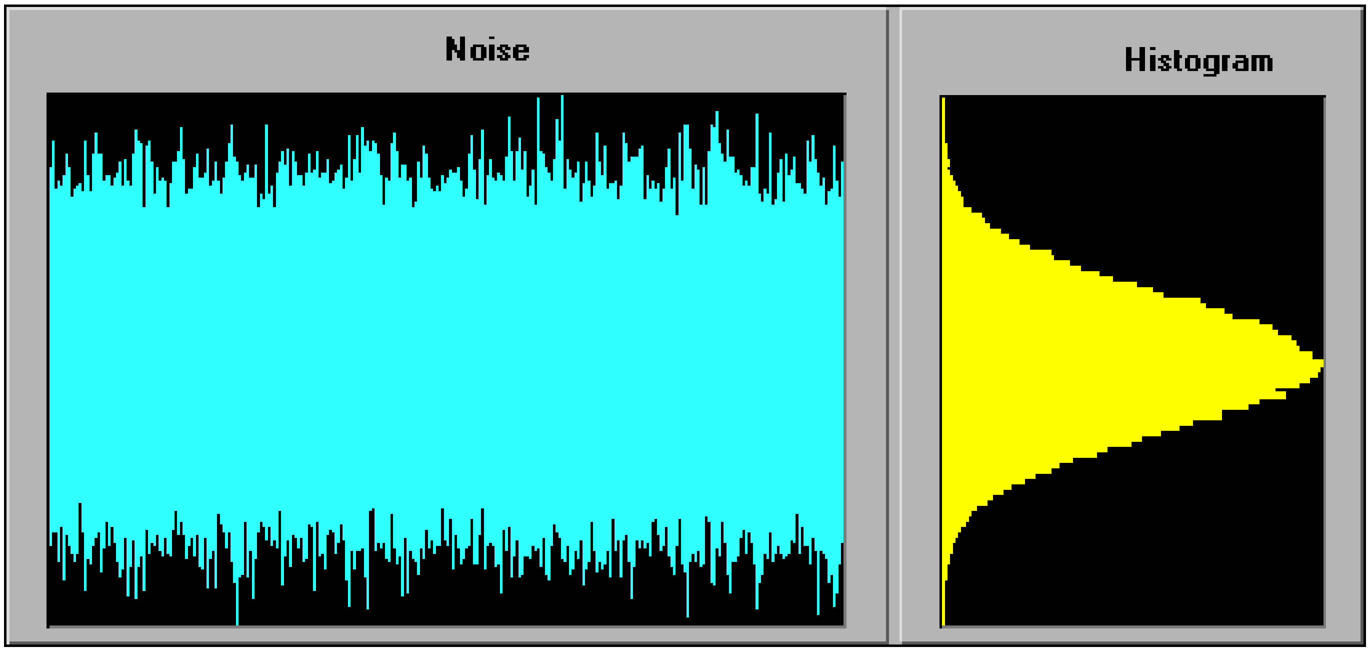
Figure 2.

Figure from https://www.edn.com/electronics-blogs/the-signal/4408242/1-f-Noise-the-flickering-candle-

Probability distribution of white noise

White noise passed through a filter generally has a *Gaussian* probability distribution. The blue trace shows white noise voltage versus time. If the output is sorted into bins according to voltage, the result is the yellow histogram. It shows a Gaussian distribution with zero mean.

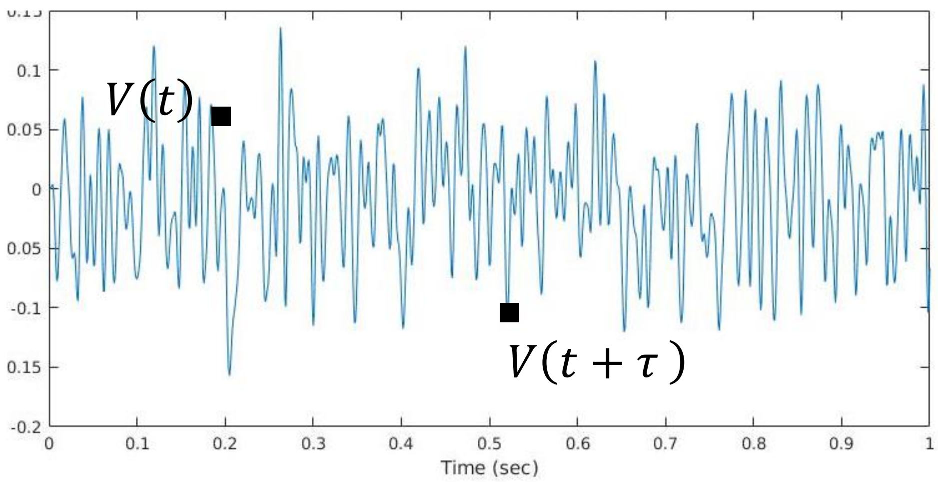
$$Prob(V) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{V^2}{2\sigma^2}} \qquad \langle V^2 \rangle = \int_{-\infty}^{\infty} V^2 Prob(V)\, dV = \sigma^2 = 4k_B TR\, \Delta f$$



From http://zone.ni.com/reference/en-XX/help/370051T-01/cvi/libref/analysisconcepts/noise_generation/

Correlation functions

A noise voltage can't be Fourier analyzed in the conventional sense since it's random and goes on forever. But we *can* say something about random signals at different points in time. That involves the *autocorrelation function*, defined as,

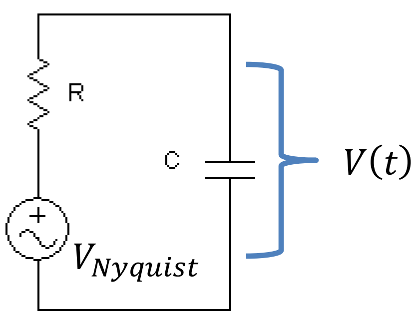


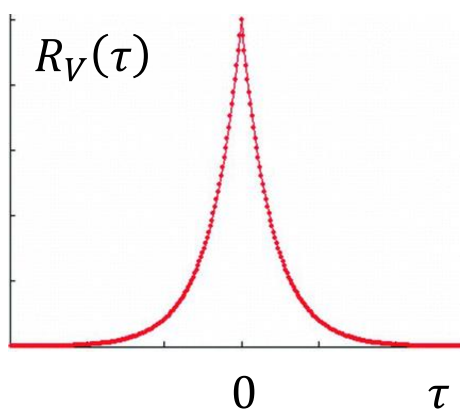
$$R_V(\tau) = \langle V(t+\tau)\, V(t)\rangle = \langle V(\tau)\, V(0)\rangle$$

This is the ensemble average of the product of a random voltage at two *different* times. If the noise is stationary then the average depends *only* on the time τ between the two measurements.

There are other basic properties of autocorrelation functions. If $\tau = 0$ then $R_V(0) = \langle V(t)\, V(t)\rangle = \langle V^2\rangle \neq 0$. On the other hand, when $\tau \to \infty$ there should be no correlation between the two voltages so $R_V(\infty) = 0$. It also shouldn't make any difference whether $\tau > 0$ or $\tau < 0$. Therefore the autocorrelation function for random noise generally obeys,

$$R_V(\tau) < R_V(0) \qquad , \qquad R_V(-\tau) = R_V(\tau)$$





These properties make $R_V(\tau)$ a mathematically well-behaved function. For example, consider an RC low pass filter where the source of EMF is the Nyquist noise voltage of the resistor. We'll show that the autocorrelation of the output across the capacitor is,

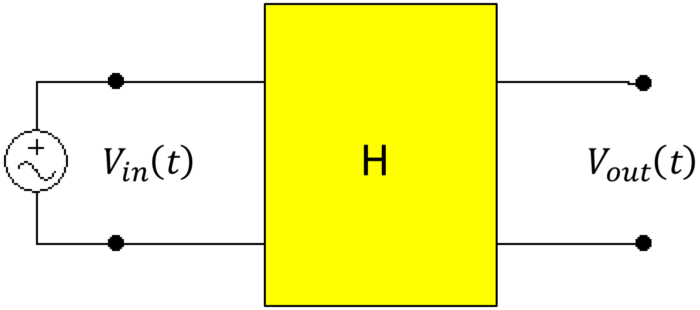
$$R_V(\tau) = \langle V^2\rangle\, e^{-|\tau|/RC}$$

Voltages across the capacitor are correlated in time because the capacitor has a memory and it cannot charge or discharge instantaneously. See F. Reif, *Fundamentals of Statistical and Thermal Physics* for a much more complete discussion.

Output from a noisy circuit

We want to find the spectral density from a circuit that has noisy elements like resistors. Consider a linear circuit with a transfer function *H* that's driven by a noise generator $V_{in}(t)$. For example, the transfer function of the simple RC low pass filter is given by,

$$H(\omega) = \frac{1}{1 + i\,\omega RC}$$



In general, for *any* linear circuit we have,

$$\hat{V}_{out}(\omega) = H(\omega)\,\hat{V}_{in}(\omega)$$

There's a theorem from Fourier analysis that if a function is the product of two Fourier transforms then in the time domain it obeys,

$$V_{out}(t) = \int_{-\infty}^{\infty} H(t')\,V_{in}(t - t')dt' \qquad H(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\,H(\omega)\,d\omega$$

The integral on the left is called a *convolution*. For example, $H(t)$ for the low pass filter is $H(t) = \theta(t)\,e^{-t/RC}/RC$ where is the unit step function. Engineers refer to $H(t)$ as the *impulse response function*.

Next, we need a theorem due to Wiener and Khinchine stating that the spectral density and the autocorrelation function are a Fourier transform pair,

$$J(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} R_V(\tau)e^{-i\omega\tau}\,d\tau \qquad R_V(\tau) = \int_{-\infty}^{+\infty} e^{i\omega\tau}\,J(\omega)\,d\omega$$

$J(\omega)$ is defined for both positive and negative frequencies. However, since R_V is an *even* function of τ then $J(\omega) = J(-\omega)$ so it's simply related to the spectral density *S* introduced earlier ,

$$\langle V^2 \rangle = R(0) = \int_{-\infty}^{+\infty} J(\omega)\,d\omega = \int_{0}^{+\infty} 2\,J(\omega)\,d\omega = \int_{0}^{+\infty} S(f)\,df \qquad S(f) \equiv 4\pi\,J(\omega)$$

For the time being it's more convenient to use $J(\omega)$ since integrals over all frequency are easier.

With this connection, we need the autocorrelation function of the output, after which we'll take its Fourier transform. $V_{in}(t)$ and $V_{out}(t)$ are both random functions. The autocorrelation function is now given by,

$$R_{out}(\tau) = \langle V_{out}(\tau)\, V_{out}(0)\rangle = \left\langle \int_{-\infty}^{\infty} H(t')\, V_{in}(\tau - t')dt' \int_{-\infty}^{\infty} H(t'')\, V_{in}(-t'')dt'' \right\rangle$$

Now take the Fourier transform and use the fact that ensemble averages and integrals commute,

$$J_{out}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\tau}\, R_{out}(\tau)d\tau = \frac{1}{2\pi}\left\langle \int_{-\infty}^{\infty} e^{-i\omega\tau}\, d\tau \int_{-\infty}^{\infty} H(t')\, V_{in}(\tau - t')dt' \int_{-\infty}^{\infty} H(t'')\, V_{in}(-t'')dt'' \right\rangle$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\tau}\, H(t')H(t'')\,\langle\, V_{in}(\tau - t')\, V_{in}(-t'')\,\rangle\, d\tau\, dt' dt''$$

But the correlation function only depends on the *difference* in times so define,

$$u \equiv (\tau - t') - (-t'') = \tau - t' - t''$$

$$J_{out}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega u}\, e^{-i\omega t'} e^{i\omega t''} H(t')H(t'')\,\langle\, V_{in}(u)\, V_{in}(0)\,\rangle\, du\, dt' dt''$$

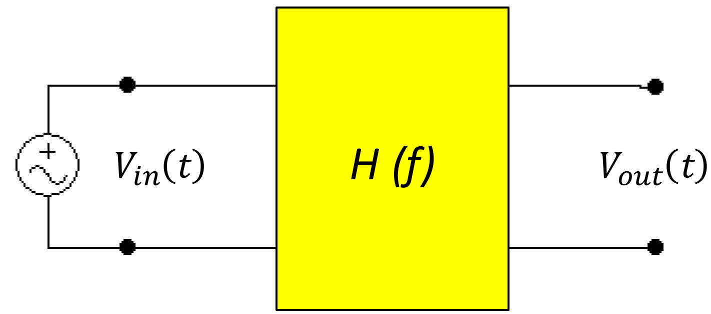
This now becomes the product of three integrals,

$$J_{out}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t'} H(t')\, dt' \int_{-\infty}^{\infty} e^{+i\omega t''} H(t'')dt'' \frac{1}{2\pi}\int_{-\infty}^{\infty} \langle V_{in}(u)\, V_{in}(0)\rangle\; e^{-i\omega u}\, du$$

$$= H(\omega)H(-\omega)\, J_{in}(\omega) \; = H(\omega)H^{*}(\omega)\, J_{in}(\omega) = |H(\omega)|^2\;\, J_{in}(\omega)$$

The final result is easy to remember,

$$J_{out}(\omega) = |H(\omega)|^2 \; J_{in}(\omega)$$

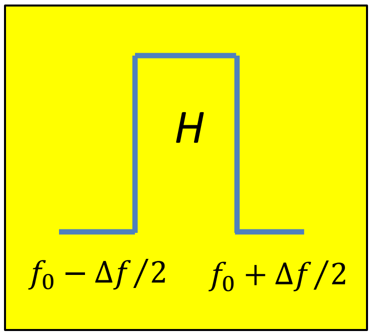


Alternatively, we can write this in terms of the one-sided spectral densities,

$$S_{out}(f) = |H(f)|^2 \; S_{in}(f)$$

1. Square filter. Consider some noise source with a spectral density $S_{in}(f)$. Now send it through a filter with a transfer function,

$$H(f) = 1 \quad (f_0 - \Delta f/2 < f < f_0 + \Delta f/2) \quad , \qquad H(f) = 0 \quad otherwise$$



The mean squared noise on the output is given by,

$$\langle V_{out}^2 \rangle = R_{out}(0) = \int_0^\infty S_{out}(f)\, df = \int_0^\infty |H(f)|^2\, S_{in}(f)\, df = \int_{f_0 - \Delta f/2}^{f_0 \Delta f/2} S_{in}(f)\, df$$

If $\Delta f \ll f_0$ then, $\quad \langle V_{out}^2 \rangle = S_{in}(f_0)\, \Delta f$

Again, this is the way to measure a power spectral density. Send the noisy waveform through a narrow band filter and then measure the mean-squared output voltage.

2. RC-filtered Nyquist noise. Let's revisit the *RC* circuit with the noisy resistor. If the generator is the Nyquist noise of the resistor then,

$$S_{out}(f)\, df = |H(f)|^2\, S_{in}(f)\, df = \frac{4 k_B T R\, df}{1 + (\omega RC)^2}$$

Imagine you could measure this noise with an instrument that had infinite bandwidth and responded the same at all frequencies $(H(f) = 1)$. There is no such instrument, of course, but if there were, that instrument would measure a *total* means-squared output noise of,

$$\langle V_{out}^2 \rangle = \int_0^{\infty} S_{out}(f)\, df = \int_0^{\infty} \frac{4k_B TR\, df}{1+(\omega RC)^2} = \frac{k_B T}{C} \quad \rightarrow \quad \left\langle \frac{1}{2} C\, V_{out}^2 \right\rangle = \frac{k_B T}{2}$$

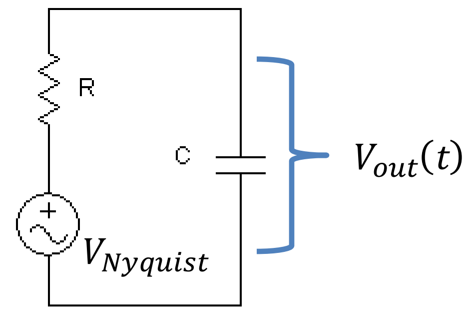
The result is exactly what we could have predicted from the equipartition theorem of statistical mechanics: In thermal equilibrium, each quadratic term in the energy of a classical system has an average energy of $k_B T/2$.

Finally, knowing $\langle V_{out}^2 \rangle$ we can go back and find the pre-factor for the autocorrelation function for the *RC* filter driven by Nyquist noise,

$$R_{out}(\tau) = \langle V_{out}^2 \rangle\, e^{-|\tau|/RC} = \frac{k_B T}{C}\, e^{-|\tau|/RC}$$

$$J_{out}(\omega) = \frac{1}{\pi} \frac{k_B TR}{1+(\omega RC)^2}$$

$$S_{out}(f) = 4\pi\, J(\omega) = \frac{4\, k_B TR}{1+(2\pi f RC)^2}$$



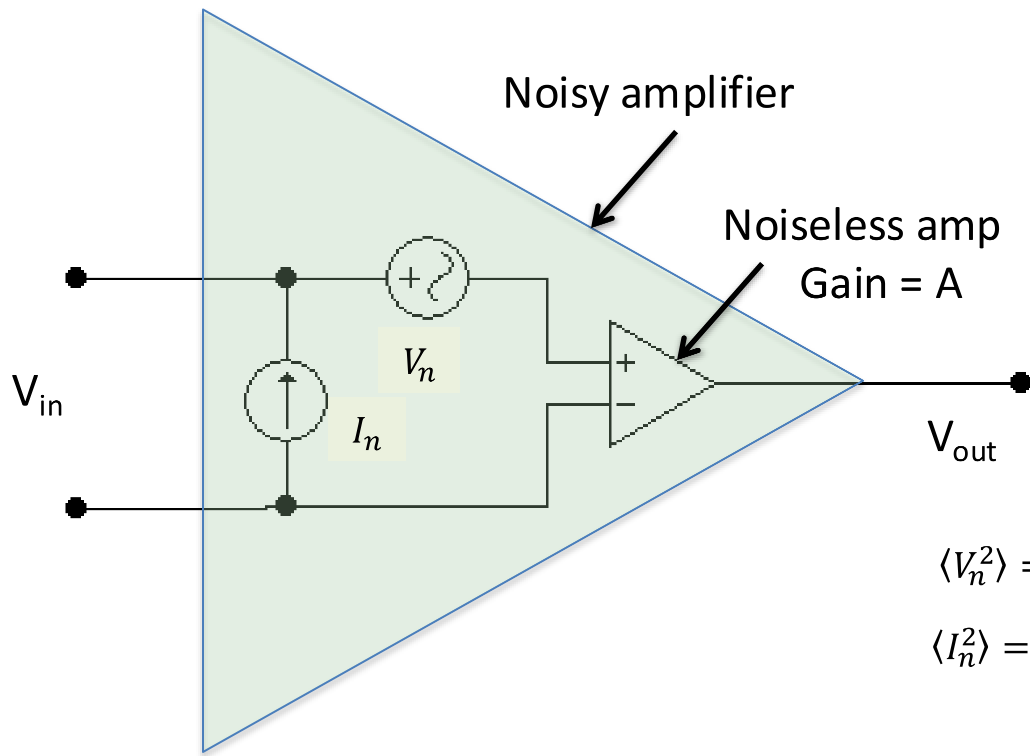
The correlation function $R_{out}(\tau)$ for *RC* filtered Nyquist noise has, not surprisingly, a correlation time *RC*. What about the Nyquist noise itself?

$$R_{Nyquist}(\tau) = \int_{-\infty}^{+\infty} e^{i\omega\tau}\, J_{Nyquist}(\omega)\, d\omega = \frac{k_B TR}{\pi} \int_{-\infty}^{+\infty} e^{i\omega\tau}\, d\omega = 2k_B TR\, \delta(\tau)$$

Nyquist noise is white so its spectral density is constant as a function of frequency. Its correlation function is therefore a *delta function* in time. Noise with a white spectrum has no correlation from one moment to the next.

Noise in amplifiers

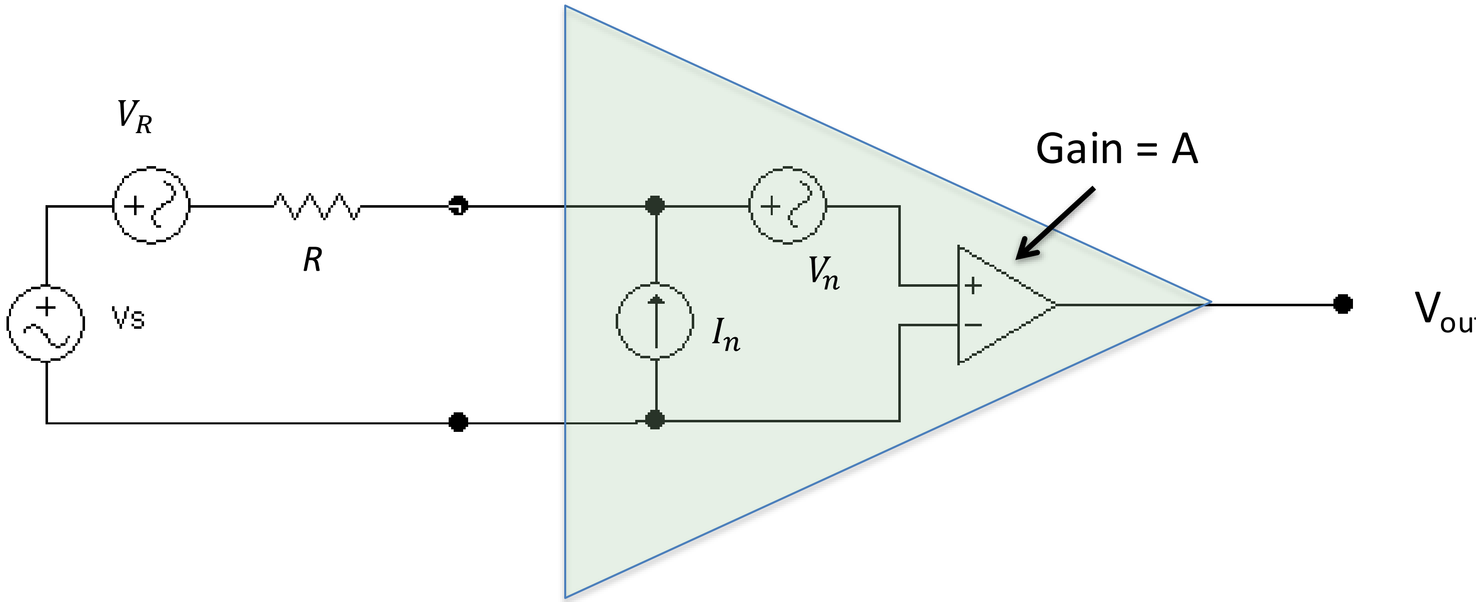
Next, we'd like to see how to treat multiple noise sources in circuits. Amplifiers are a useful example. Even the simplest amp has many resistors and transistors so it quickly becomes very complicated to predict the noise with any reliability. Instead, engineers use a model that takes account of the two irreducible noise generators – voltage and current noise. We then connect the amplifier to external components and calculate the quantity that matters – the signal to noise ratio S/N.



$$\langle V_n^2 \rangle = e_n^2 \, \Delta f$$

$$\langle I_n^2 \rangle = i_n^2 \, \Delta f$$

The model consists of a real (noisy) amp, inside of which is an ideal, noiseless amp of gain *A*. All the amplifier noise is accounted for by the two sources denoted V_n and I_n. V_n is a voltage noise generator with spectral density $S_e = e_n^2$ and I_n is a current noise generator with spectral density $S_i = i_n^2$. The units are $e_n \sim Volt/\sqrt{Hz}$ and $i_n \sim Amp/\sqrt{Hz}$. e_n and i_n represent the sum total of all the noisy amplifier circuitry. Roughly speaking they account for Nyquist noise and shot noise in the amplifier circuitry but their values depend on the specific amplifier. For simplicity I've assumed that the ideal amp has infinite input impedance and the gain *A* is just a constant. After you go through this simple case you can always add a finite input impedance R_{in} and make the gain complex A and frequency dependent.



Now hook the amplifier up to a deterministic signal source V_S which, like all signal sources, has some output impedance *R*. We must include the Nyquist noise of *R* so there are 3 noise generators corresponding to noise voltages V_R , V_n and a noise current I_n. Now use superposition.

1. Turn off all the noise sources. V_R , V_n become *short* circuits and I_n becomes an *open* circuit. The ideal amp has infinite input impedance so no current flows through *R* and the output is just,

$$Sig = Output\ signal = A\, V_S$$

2. Now turn off V_S, V_n , I_n . Then the instantaneous output from the Nyquist noise generator is,

$$V_{out}^{R} = A\, V_R$$

Where V_R is the random Nyquist voltage noise from *R*.

3. Turn off everything except V_n. Its output contribution is,

$$V_{out}^{n} = A\, V_n$$

4. Finally, turn off everything except I_n . The ideal amplifier has infinite input impedance so I_n goes back through the resistor *R* and generates a noise voltage $I_n R_S$. This voltage is multiplied by *A* to produce an output voltage of,

$$V_{out}^{I} = A\, I_n R_S$$

The *total* instantaneous output noise voltage is,

$$V_{out} = A(V_R + V_n + I_n R_S)$$

This quantity is random so look at its autocorrelation function,

$$R_{out}(\tau) = \langle V_{out}(t+\tau) V_{out}(t) \rangle$$

$$R_{out}(\tau) = A^2 \langle V_R(t+\tau) V_R(t) \rangle + A^2 \langle V_n(t+\tau) V_n(t) \rangle + A^2 \langle R_S I_n(t+\tau)\ R_S I_n(t) \rangle + \boldsymbol{Cross\ terms}$$

The cross terms are things like $\langle V_R(t+\tau) V_n(t) \rangle$ or $\langle V_n(t+\tau) R_S I_n(t) \rangle$. But these are all zero. There is no reason for the resistor Nyquist noise to be correlated with the amplifier current noise, for example. So *all* the noise comes from the autocorrelation functions of the 3 independent noise sources:

$$R_{out}(\tau) = A^2 \langle V_R(t+\tau) V_R(t) \rangle + A^2 \langle V_n(t+\tau) V_n(t) \rangle + A^2 R_S^2 \langle I_n(t+\tau)\ I_n(t) \rangle$$

This implies that the power spectra of the individual , uncorrelated noise sources just add. The mean-squared output noise in a bandwidth Δf is then given by,

$$\langle V_{out}^2 \rangle = S_{out}(f)\, \Delta f = A^2 \left[\, 4Rk_B T + e_n^2 + R_S^2\, i_n^2 \right] \Delta f$$

Suppose the signal source has a mean-squared amplitude V_S^2. It gets multiplied by the same factor of A^2 as the noise. The output signal to noise ratio is given by,

$$\frac{Sig}{N} = \sqrt{\frac{V_S^2}{\langle V_{out}^2 \rangle}} = \frac{V_S}{\sqrt{[4Rk_B T + e_n^2 + R_S^2\, i_n^2]\, \Delta f}}$$

Examples

Typical values for the amplifier noise sources are $e_n \approx 10^{-9}$ - 10^{-8} V/(Hz)$^{1/2}$ and $i_n \approx 10^{-12} – 10^{-14}$ A/(Hz)$^{1/2}$. Let's look at two examples using the circuit just analyzed.

Case 1. e_n = 10 nV/(Hz)$^{1/2}$ i_n = 1 pA/(Hz)$^{1/2}$ R = 100 Ω. The 3 noise sources give,

$$e_n^2 = 10^{-16} \quad i_n^2 R_S^2 = 10^{-20} \quad 4Rk_BT = 10^{-20}$$

In this case the amplifier voltage noise e_n dominates. Suppose we look at the output voltage with an instrument whose bandwidth $\Delta f = 10^4\ Hz$. The RMS voltage noise will be,

$$V_{RMS} = \sqrt{[4Rk_BT + e_n^2 + R_S^2\ i_n^2]\ \Delta f} \approx 10^{-6}\ V$$

In order to see a signal we would require $V_S > 10^{-6}$ V .

Case 2. e_n = 10 nV/(Hz)$^{1/2}$ i_n = 1 pA/(Hz)$^{1/2}$ but now suppose the source resistor R = 1 MΩ. The noise contributions are now,

$$e_n^2 = 10^{-16} \quad i_n^2 R_S^2 = 10^{-12} \quad 4Rk_BT = 10^{-14}$$

This time, the current noise of the amplifier flowing through the source resistor R dominates the noise. With the same $\Delta f = 10^4\ Hz$ we have,

$$V_{RMS} = \sqrt{[4Rk_BT + e_n^2 + R_S^2\ i_n^2]\ \Delta f} \approx 10^{-4}\ V$$

This time the signal voltage needs to be greater than $10^{-4}\ V$ to be observable.

From these examples you can see that noise depends not only on the amplifier itself but by the source resistance R of the circuit driving it. **By far, the most common way to reduce noise is to decrease the bandwidth Δf .**

Rules for circuit noise analysis

Now that you've seen how it goes you don't need all the correlation functions and Fourier transforms. For uncorrelated noise sources, the rules are basically,

a. Using superposition and phasors, find the the transfer function *H*(f) for each noise source, treated just as if it were an ordinary harmonic source.

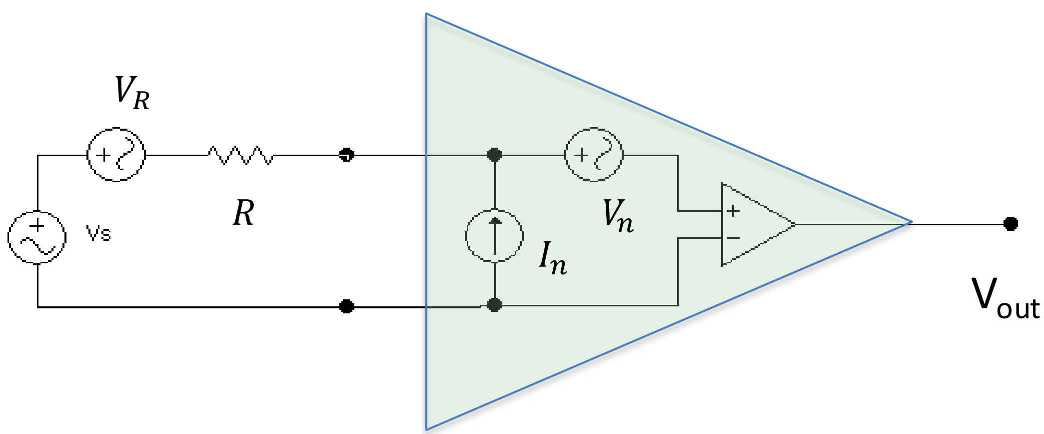
b. Convert each output noise contribution into its corresponding spectral density using $S_{out}(f) = |H(f)|^2 \; S_{in}(f)$.

c. Add the spectral densities to get the total mean-squared noise on the output in a given bandwidth Δf .

Noise Figure

It's often useful to have a figure of merit for amplifier noise. Referring to the amplifier circuit we just analyzed, take the ratio of the total output noise to the Nyquist noise from *R* alone:

$$F = \frac{\langle V_{out}^2 \rangle}{\langle V_{Nyquist}^2 \rangle} = \frac{[4Rk_BT + e_n^2 + R_S^2 \, i_n^2] \, \Delta f}{4Rk_BT \, \Delta f} = 1 + \frac{e_n^2}{4Rk_BT} + \frac{R}{4k_BT} i_n^2$$



Note that the gain and bandwidth cancel out in this ratio. It is generally quoted in decibels and called the *noise figure (NF),*

$$NF(dB) = 10 \; log_{10} F$$

A noiseless amplifier would have *NF* = 0 dB. *NF* depends on the source resistance *R*. In high frequency amps *NF* is typically quoted for *R* = 50 Ohms. Low noise amps for cellular communications often have *NF* < 1 dB.

The presence of *R* in the noise figure can lead to confusion. For example, we can minimize *NF* with respect to *R* and obtain,

$$\frac{dF}{dR} = 0 \quad \rightarrow R = \frac{e_n}{i_n}$$

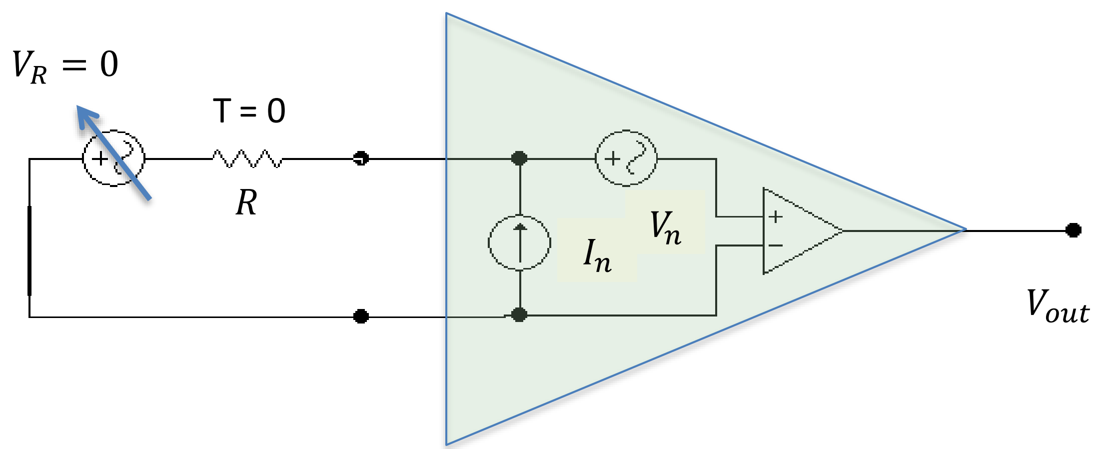
If *R* were smaller than e_n/i_n, you might be tempted to add resistance until the total source resistance = e_n/i_n. However, increasing the source resistance just adds Nyquist and I^2R current noise to the total noise. In order satisfy the above equation, you would need to transform the resistance *R* using a *noiseless* transformer. Sometimes this works, particularly for low frequencies, but transformers are heavy, complicated and often not worth the effort.

Noise temperature

Noise temperature is another figure of merit for amplifiers. To see how it works consider 2 thought experiments.

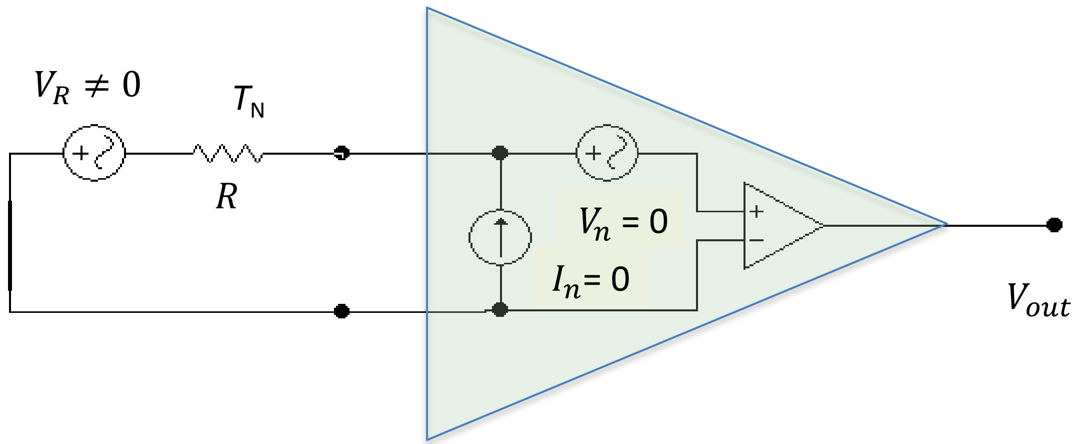
Case 1. Take the circuit below, imagine cooling just *R* down to $T = 0$ but keeping its resistance the same. In that case, the Nyquist noise V_R vanishes. The noise from the amp (gain = *A*) in a bandwidth Δf is,

$$\langle V_{out}^2 \rangle = A^2 [\, e_n^2 + R^2 i_n^2 \,] \, \Delta f \qquad T = 0$$



Case 2. Instead, grab a perfectly *noiseless* amp with the same gain and connect it to R. But now, heat *R* up to some temperature T_N. Now the total noise from the amplifier is *just* the amplified Nyquist noise from R:

$$e_n\,, i_n = 0 \quad \rightarrow \langle V_{out}^2 \rangle = A^2\, 4k_B T_N R\, \Delta f$$



The *noise temperature* T_N of the real (noisy) amplifier is defined by setting the noise for case 1 equal to the noise from case 2:

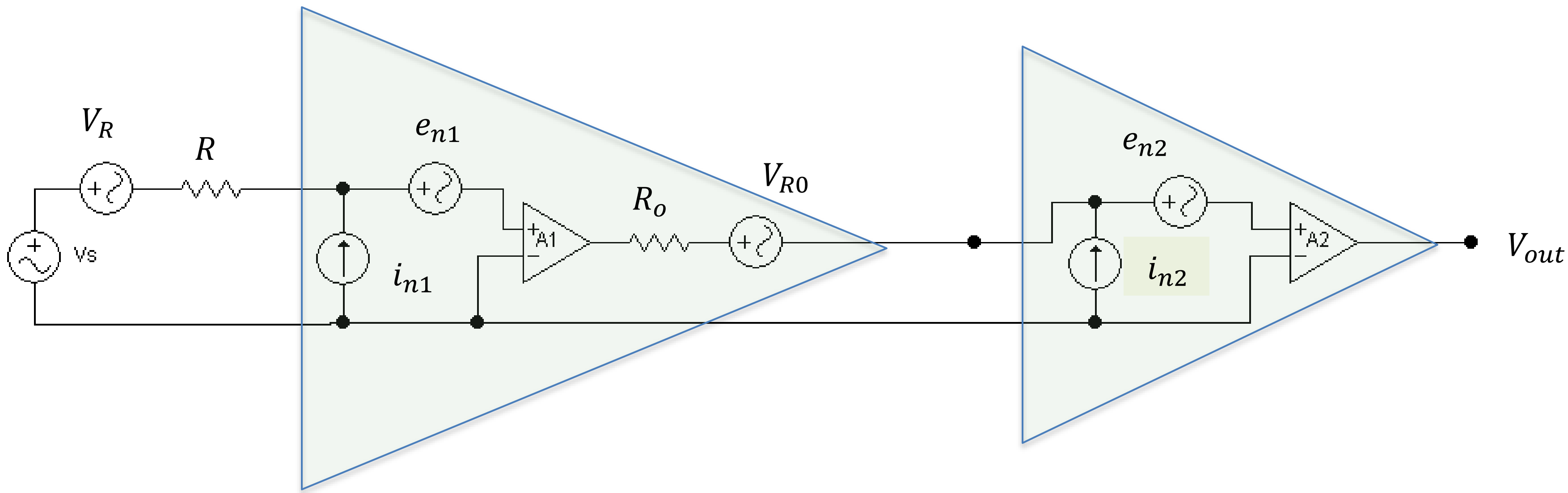
$$4k_B T_N R = e_n^2 + R^2 i_n^2$$

T_N is the temperature at which the Nyquist noise from *R alone* would equal the noise coming *only* from the amplifier itself connected to a noiseless *R.* It is *not* the actual temperature of the amplifier! In fact, very quiet preamps often have noise a temperature T_N well below the physical *T* of the amplifier. It's easy to show that noise temperature and noise figure are related:

$$NF = 10\; log_{10}\left(1 + \frac{T}{T_N}\right)$$

Importance of preamps

Low noise electronics generally involves a carefully designed low noise *preamp* as the first stage. Later stages may be designed for power output rather than low noise. To see why, consider the two-stage circuit shown below. The first amp has gain A_1 along with voltage and current noise densities e_{n1} and i_{n1}. The second stage has A_2, e_{n2} and i_{n2}. R_0 is the output impedance of the first stage, which is typically small.



Again, we can do this by superposition and then just add up the mean squared voltages from all the sources. The second amplifier stage multiplies all the voltages (noise and signal) from stage 1 and then adds noise of its own. The the output noise voltage from the chain of amplifiers is,

$$\langle V_{out}^2\rangle(noise) = A_1^2A_2^2(4Rk_BT + e_{n1}^2 + R^2i_{n1}^2)\Delta f + A_2^2(4R_0k_BT + e_{n2}^2 + R_0^2i_{n2}^2)\Delta f$$

The signal to noise ratio is given by,

$$\left(\frac{Sig}{N}\right)^2 = \frac{A_1^2A_2^2\,V_S^2}{A_1^2A_2^2(4Rk_BT + e_{n1}^2 + R^2i_{n1}^2)\Delta f + A_2^2(4R_0k_BT + e_{n2}^2 + R_0^2i_{n2}^2)\Delta f}$$

Or,

$$\left(\frac{Sig}{N}\right)^2 = \frac{V_S^2}{(4Rk_BT + e_{n1}^2 + R^2 i_{n1}^2)\Delta f + \frac{1}{A_1^2}(4R_0k_BT + e_{n2}^2 + R_0^2 i_{n2}^2)\Delta f}$$

Notice that the noise coming from the second stage is reduced by $1/A_1^2$ where A_1 is the gain from the first stage, i.e., the *preamp*. Therefore, if the preamp has high gain and low noise, it *dominates* the overall signal to noise ration of the amplifier chain. Even within the preamp itself, the noise is often dominated by the input stage transistors.