Introduction to Fluid Dynamics

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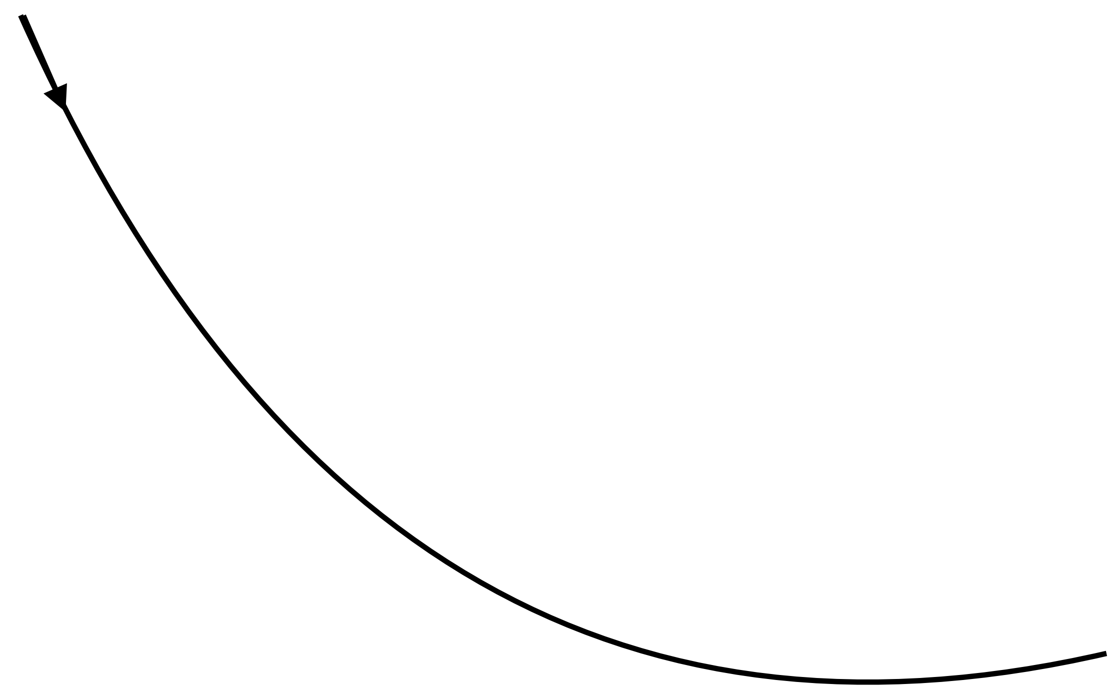
Convective Derivatives and Partial Derivatives

Partial time derivative $\frac{\partial q}{\partial t}$: rate of change of *q(t,x,y,z)* at a fixed location.

Convective time derivative $\frac{dq}{dt}$: rate of change of *q* along a path.



$$\frac{dq}{dt} = \frac{\partial q}{\partial t} + \frac{\partial q}{\partial x}\frac{dx}{dt} + \frac{\partial q}{\partial y}\frac{dy}{dt} + \frac{\partial q}{\partial z}\frac{dz}{dt} = \frac{\partial q}{\partial t} + \frac{\partial q}{\partial x}v_x + \frac{\partial q}{\partial y}v_y + \frac{\partial q}{\partial z}v_z$$

$$= \frac{\partial q}{\partial t} + \vec{v} \cdot \overrightarrow{\nabla} q$$

$$\boxed{\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \overrightarrow{\nabla}}$$

Continuity Equation I

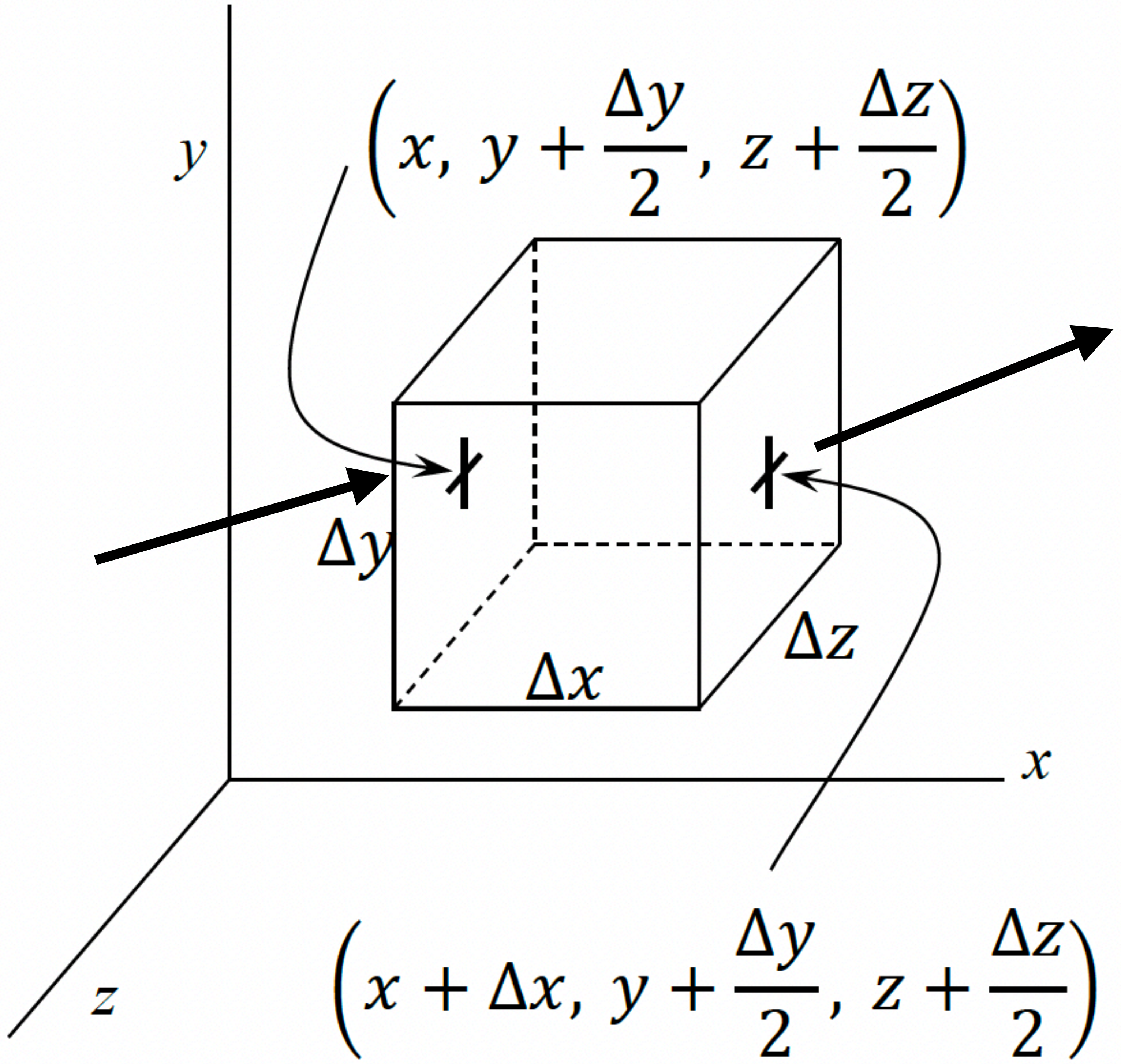
Net mass flow rate in the x-direction:

$$\Delta \dot{m}_x = \rho\left(x, y+\frac{\Delta y}{2}, z+\frac{\Delta z}{2}\right) v_x\left(x, y+\frac{\Delta y}{2}, z+\frac{\Delta z}{2}\right)\Delta y \Delta z$$

$$-\rho\left(x+\Delta x, y+\frac{\Delta y}{2}, z+\frac{\Delta z}{2}\right) v_x\left(x+\Delta x, y+\frac{\Delta y}{2}, z+\frac{\Delta z}{2}\right)\Delta y \Delta z$$

$$= -\frac{\partial}{\partial x}(\rho v_x)\Delta x \Delta y \Delta z$$

$$= -\frac{\partial}{\partial x}(\rho v_x)\Delta V$$



Continuity Equation II

Similarly, net mass flow rate in the y and z directions are

$$\Delta \dot{m}_y = -\frac{\partial}{\partial y}(\rho v_y)\Delta V \quad , \quad \Delta \dot{m}_z = -\frac{\partial}{\partial z}(\rho v_z)\Delta V$$

Total mass flowing into the volume/time is

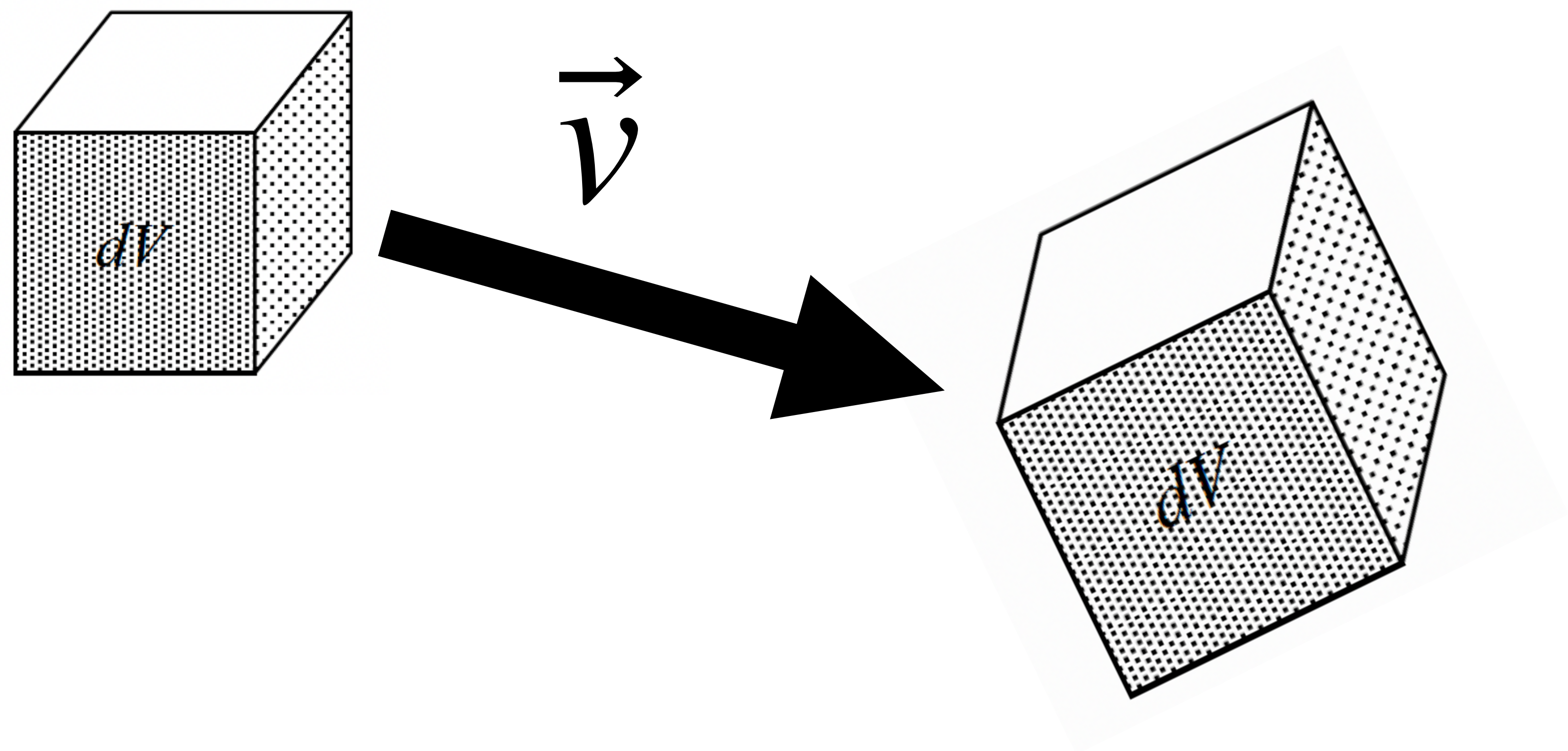
$$\Delta \dot{m} = \frac{\partial}{\partial t}(\rho \Delta V) = -\left[\frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z)\right]\Delta V = -\vec{\nabla}\cdot(\rho\vec{v})\Delta V$$

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0}$$

This is called the *continuity equation*.

Continuity Equation III

Suppose we follow the motion of the fluid.



Recall: $\dfrac{d\rho}{dt} = \dfrac{\partial \rho}{\partial t} + \vec{v} \cdot \vec{\nabla} \rho$

$$\frac{d\rho}{dt} = -\vec{\nabla} \cdot (\rho \vec{v}) + \vec{v} \cdot \vec{\nabla} \rho = -\rho \vec{\nabla} \cdot \vec{v}$$

$$\boxed{\frac{d\rho}{dt} + \rho \vec{\nabla} \cdot \vec{v} = 0}$$

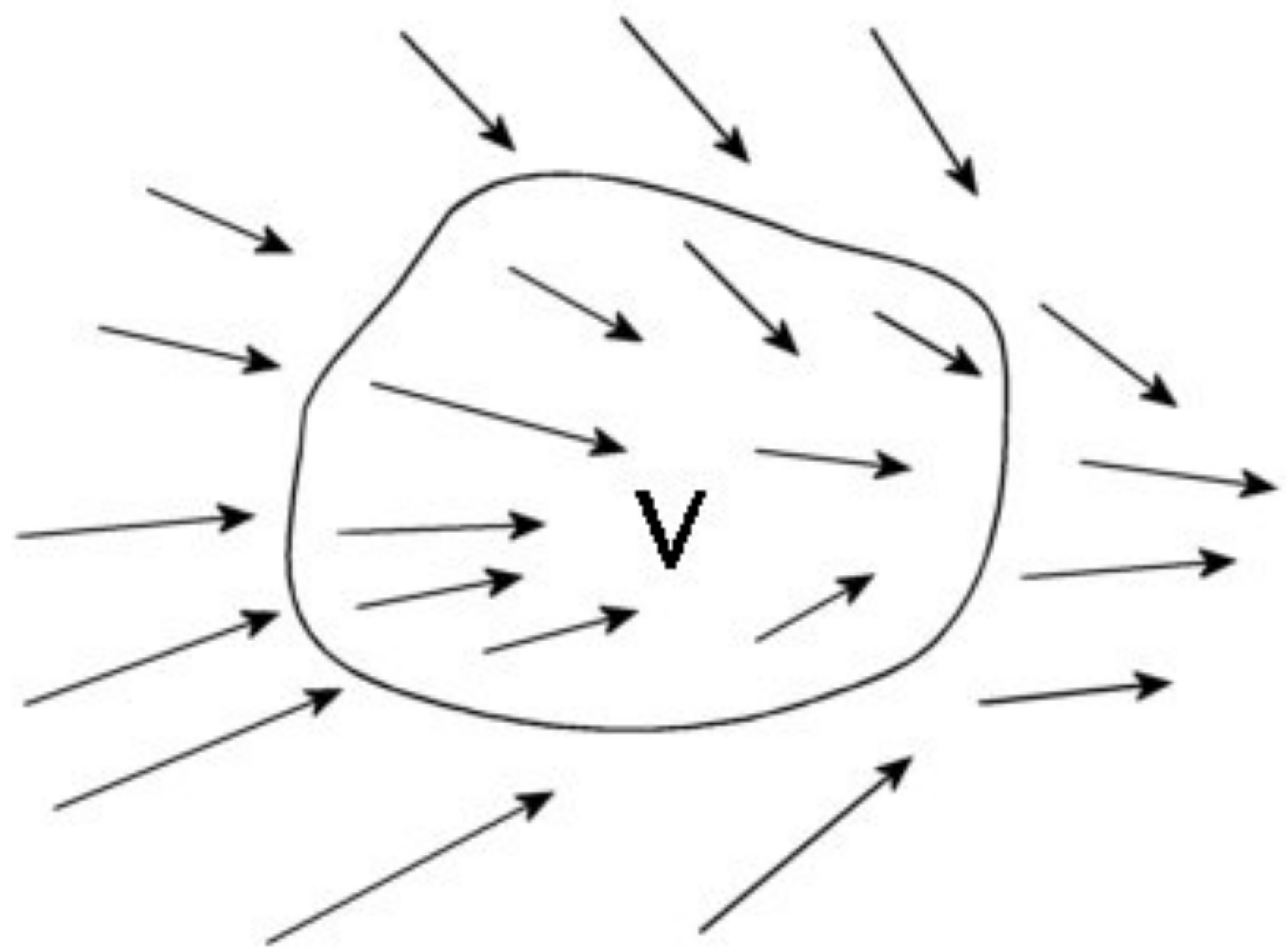
For incompressible fluid, $d\rho/dt = 0$. Hence $\vec{\nabla} \cdot \vec{v} = 0$.

Integral Form of Continuity Equation

$$M = \int_V \rho dV$$

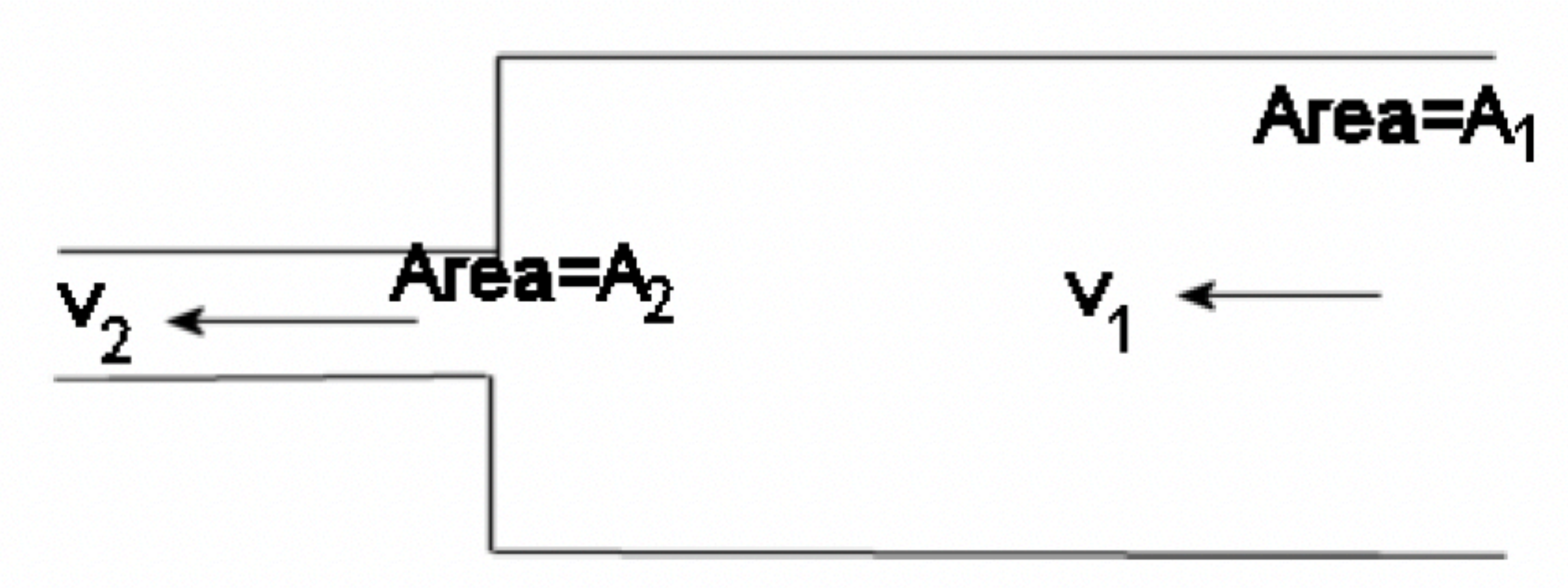
$$\frac{dM}{dt} = \int_V \frac{\partial \rho}{\partial t} dV = - \int_V \vec{\nabla} \cdot (\rho \vec{v}) dV$$

$$= - \oint_{\partial V} \rho \vec{v} \cdot d\vec{S}$$



Rate of increase in mass inside a volume *V* = net mass flow into the volume per unit time.

Example 1: Flow Tube



Consider air flowing from a tube with cross-sectional area A_1 into a region with cross-sectional area A_2.
In steady air flow, $dM/dt = 0$.

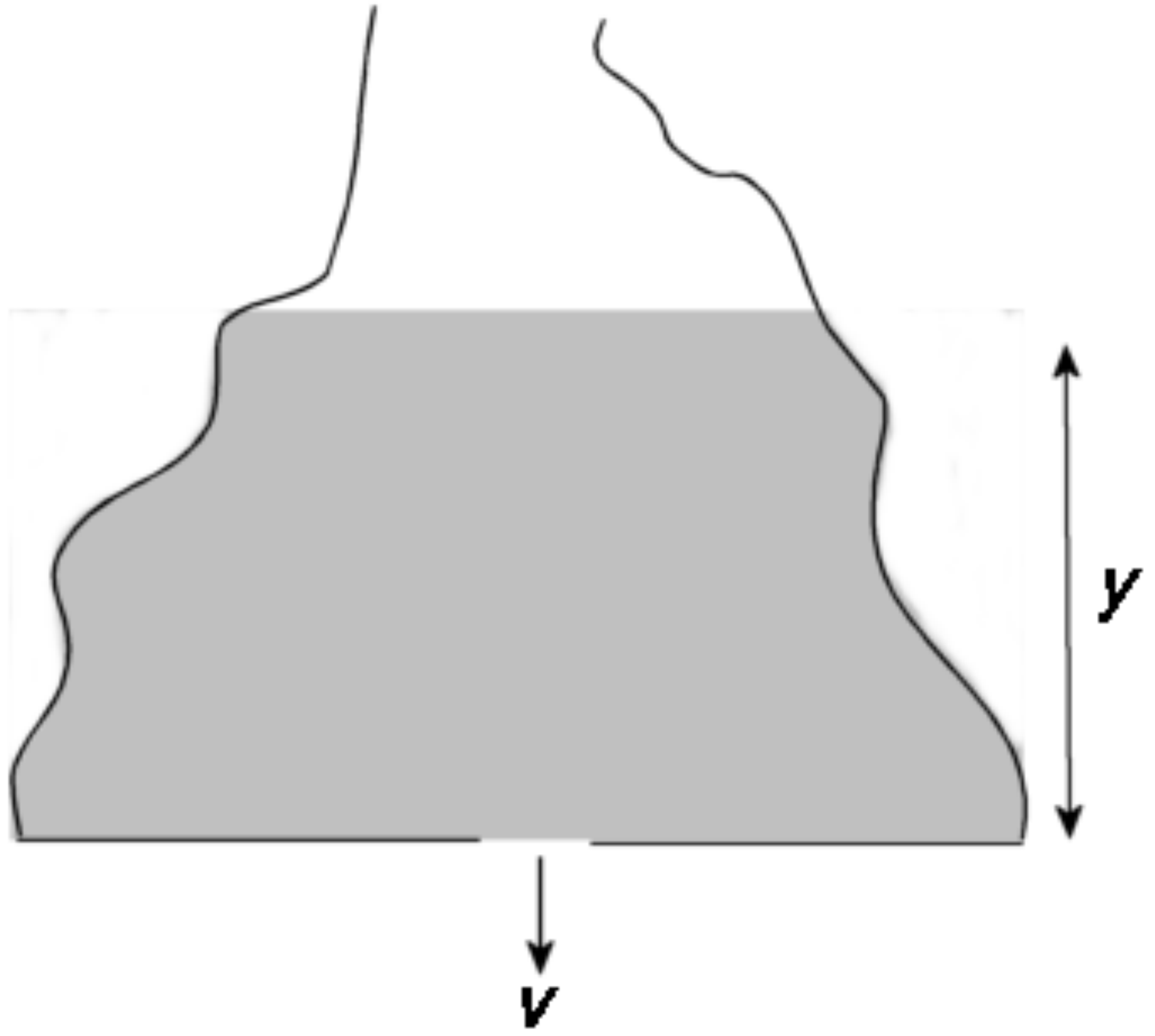
$$\rho v_1 A_1 = \rho v_2 A_2$$

$$v_2 = \frac{A_1}{A_2} v_1$$

Example 2: Water Leak

There is a small hole at the bottom of a container and water leaks out from the hole at speed *v*.

The water level *y* decreases slowly.



$$\frac{dM}{dt} = \frac{d(\rho V)}{dt} = -\rho v A_h$$

A_h: area of the hole. *V* = Volume of water inside the container.

$$\frac{dV}{dt} = A(y)\dot{y}$$

A(y): cross-sectional area at *y*

$$\Rightarrow \qquad \dot{y} = -\frac{A_h}{A(y)}v$$

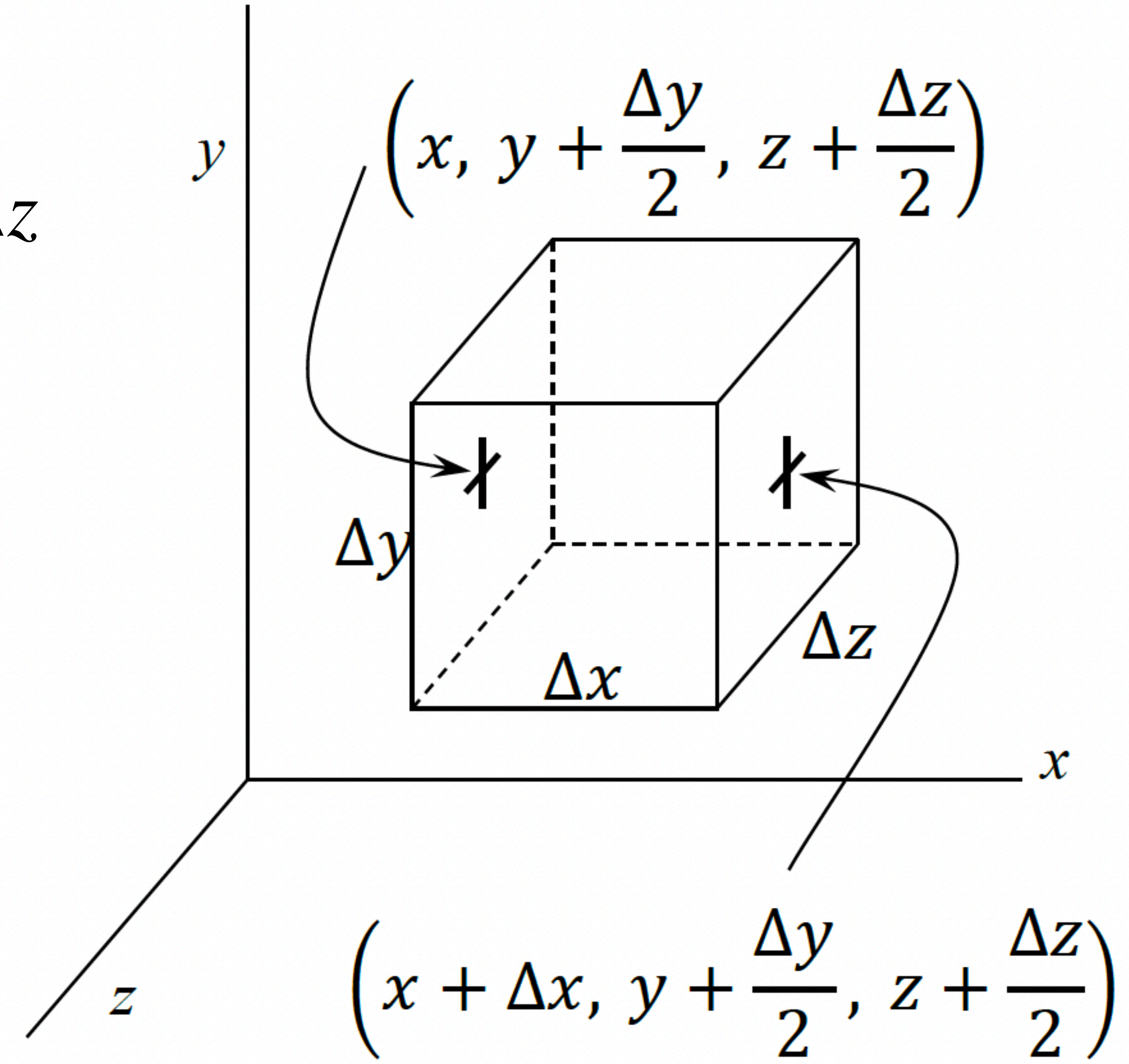
Momentum Equation

Net force associated with pressure in *x*-direction:

$$\Delta f_x = P\left(x, y+\frac{\Delta y}{2}, z+\frac{\Delta z}{2}\right)\Delta y \Delta z - P\left(x+\Delta x, y+\frac{\Delta y}{2}, z+\frac{\Delta z}{2}\right)\Delta y \Delta z$$

$$= -\frac{\partial P}{\partial x}\Delta x \Delta y \Delta z$$

$$= -\frac{\partial P}{\partial x}\Delta V$$



Similarly, $\Delta f_y = -\frac{\partial P}{\partial y}\Delta V, \quad \Delta f_z = -\frac{\partial P}{\partial z}\Delta V$

Total net force associated with pressure:

$$\Delta \vec{f} = -\left(\frac{\partial P}{\partial x}\hat{x} + \frac{\partial P}{\partial y}\hat{y} + \frac{\partial P}{\partial z}\hat{z}\right)\Delta V = -\vec{\nabla} P \Delta V$$

Momentum Equation (cont)

In addition to pressure, gravity also acts on the fluid:

$$\Delta\vec{f} = -\vec{\nabla}P\Delta V + (\rho\Delta V)\vec{g}$$

From Newton's second law:

$$(\rho\Delta V)\frac{d\vec{v}}{dt} = -\vec{\nabla}P\Delta V + \rho\vec{g}\Delta V$$

$$\boxed{\frac{d\vec{v}}{dt} = \frac{\partial\vec{v}}{\partial t} + \vec{v}\cdot\vec{\nabla}\vec{v} = -\frac{\vec{\nabla}P}{\rho} + \vec{g}}$$

This is also called *Euler's equation*.

It describes the conservation of momentum of an *ideal fluid* (i.e. without viscosity).

The Meaning of $\vec{v} \cdot \overrightarrow{\nabla} \vec{v}$

$$\vec{v} \cdot \overrightarrow{\nabla} \vec{v} = v_x \frac{\partial \vec{v}}{\partial x} + v_y \frac{\partial \vec{v}}{\partial y} + v_z \frac{\partial \vec{v}}{\partial z}$$

$$= \left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) \hat{x} + \left(v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) \hat{y} + \left(v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) \hat{z}$$

If $\vec{v}$ is represented by a row vector, $\overrightarrow{\nabla} \vec{v}$ represented by a 3×3 matrix, $\vec{v} \cdot \overrightarrow{\nabla} \vec{v}$ can be represented by a row vector by

$$\vec{v} \cdot \overrightarrow{\nabla} \vec{v} = \begin{pmatrix} v_x & v_y & v_z \end{pmatrix} \begin{pmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_y}{\partial x} & \frac{\partial v_z}{\partial x} \\ \frac{\partial v_x}{\partial y} & \frac{\partial v_y}{\partial y} & \frac{\partial v_z}{\partial y} \\ \frac{\partial v_x}{\partial z} & \frac{\partial v_y}{\partial z} & \frac{\partial v_z}{\partial z} \end{pmatrix}$$

Hydrostatics

Momentum equation: $$\frac{d\vec{v}}{dt} = -\frac{\vec{\nabla} P}{\rho} + \vec{g}$$

Hydrostatics: $\vec{v} = 0 \quad \Rightarrow \quad \vec{\nabla} P = \rho \vec{g}$

Pressure gradient is parallel to $\vec{g}$ ⇒ surface of constant *P* (isobar) is perpendicular to $\vec{g}$.

$$0 = \vec{\nabla} \times \vec{\nabla} P = \vec{\nabla} \rho \times \vec{g}$$

⇒ density gradient is parallel to $\vec{g}$ ⇒ surface of constant ρ is perpendicular to $\vec{g}$.

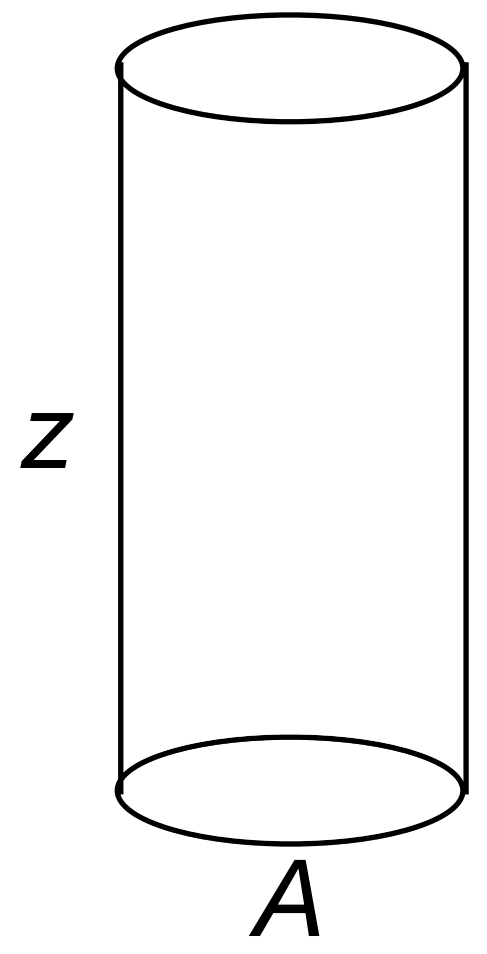
Let $\vec{g} = g\hat{z}$ ($\hat{z}$ points downward), $P = P(z), \rho = \rho(z)$.

$$\vec{\nabla} P = \frac{dP}{dz}\hat{z} = \rho g \hat{z}$$

Hydrostatics (cont)

$$\frac{dP}{dz} = \rho g$$

$$P(z) = \int \rho(z) g dz$$



Consider a cylinder with cross-sectional area *A* and height *z.*

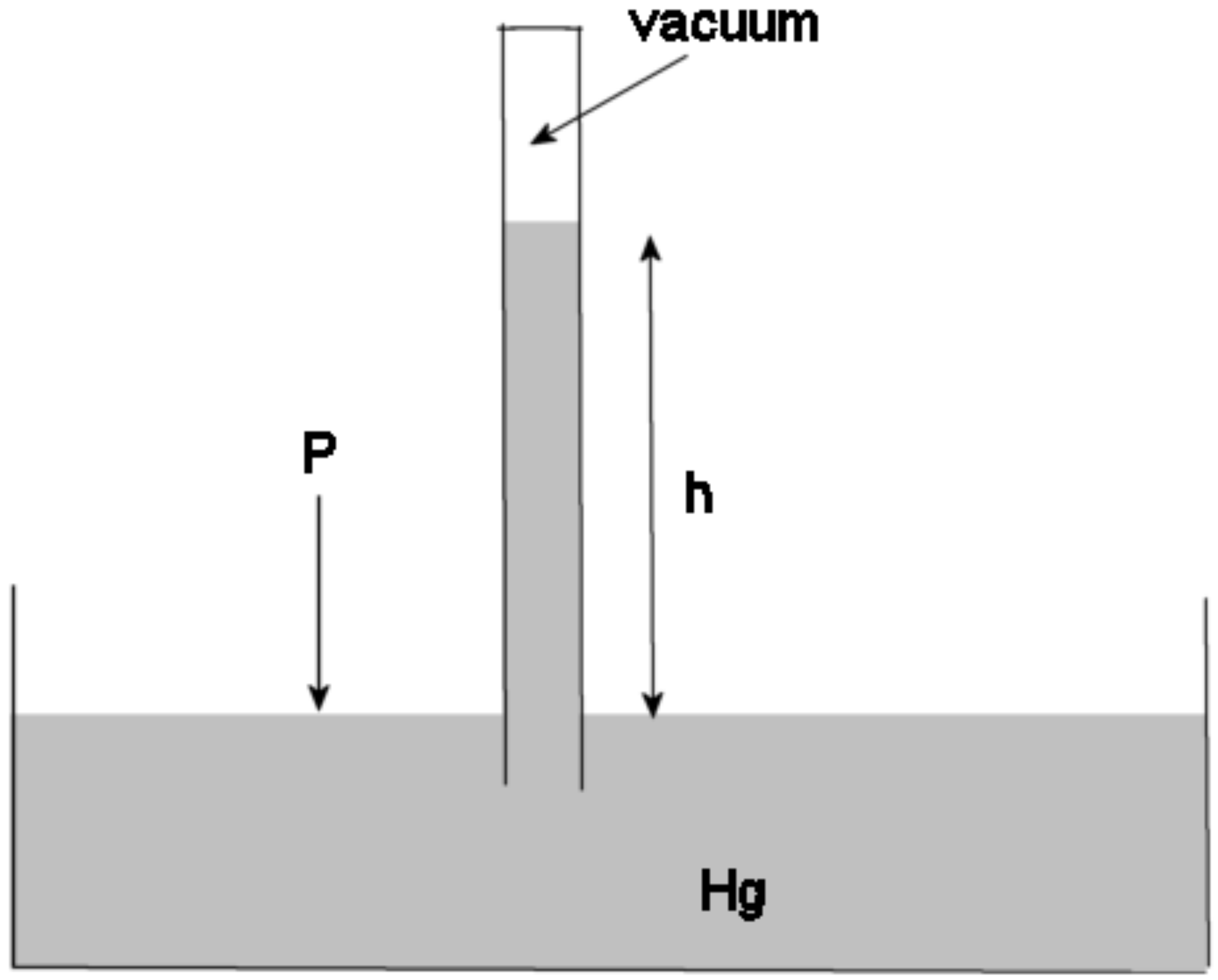
$$P(z) = \frac{1}{A}\left(\int \rho(z) A dz\right) g = \frac{M_f(z) g}{A}$$

Pressure at depth *z* is the weight of the fluid per unit area above *z*.

For incompressible fluid, $\rho(z) = \rho$ is constant,

$$P(z) = \rho g z$$

Mercury Barometer



$$P = \rho_{\mathrm{Hg}} g h$$

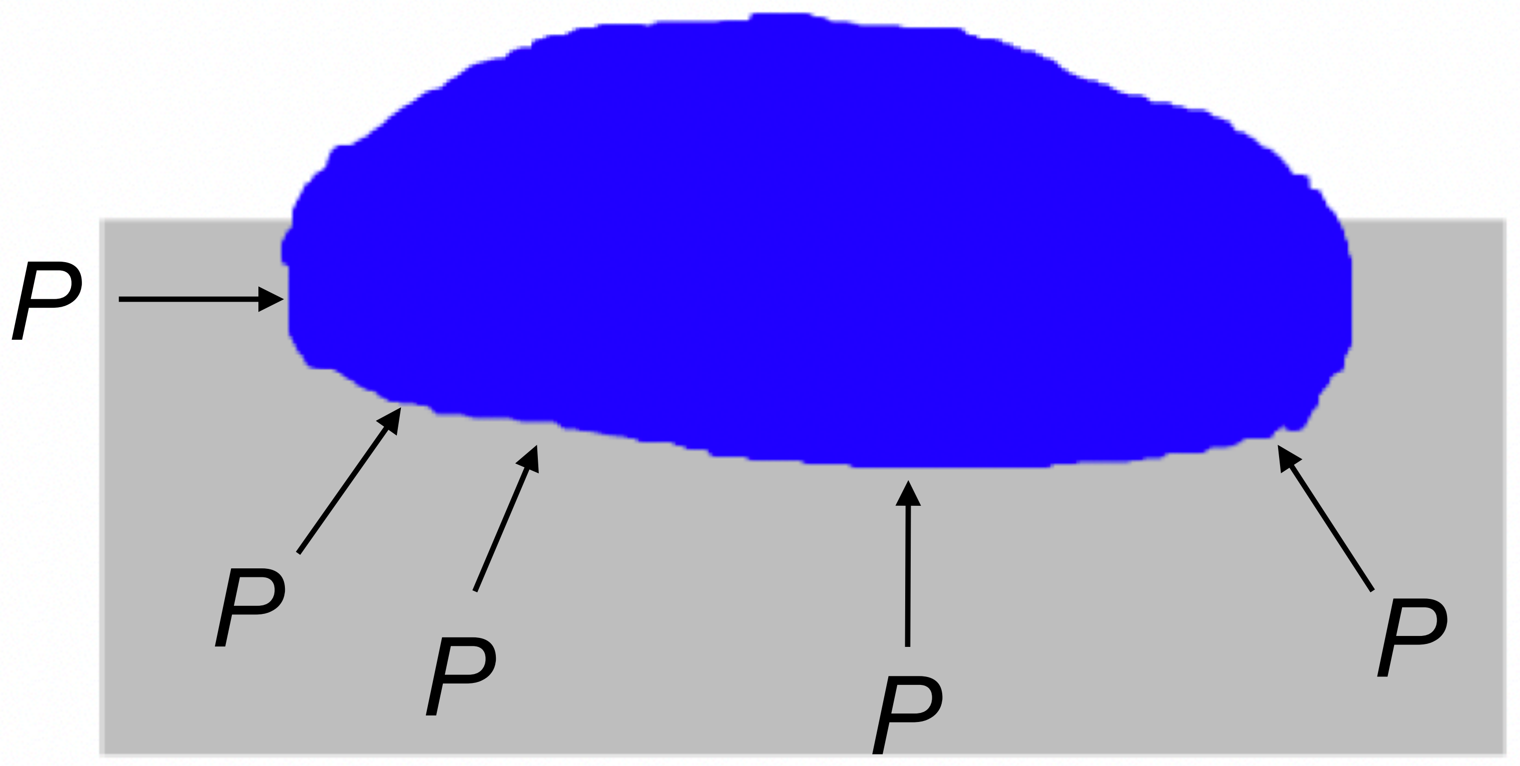
Standard atmospheric pressure = 101kPa ≈760 mmHg

Archimedes' Principle

Consider an object floating stationary in a fluid.

Buoyant force acting on the object:

$$\vec{F}_{\text{buoy}} = -\int_{\text{surface}} P d\vec{A}$$

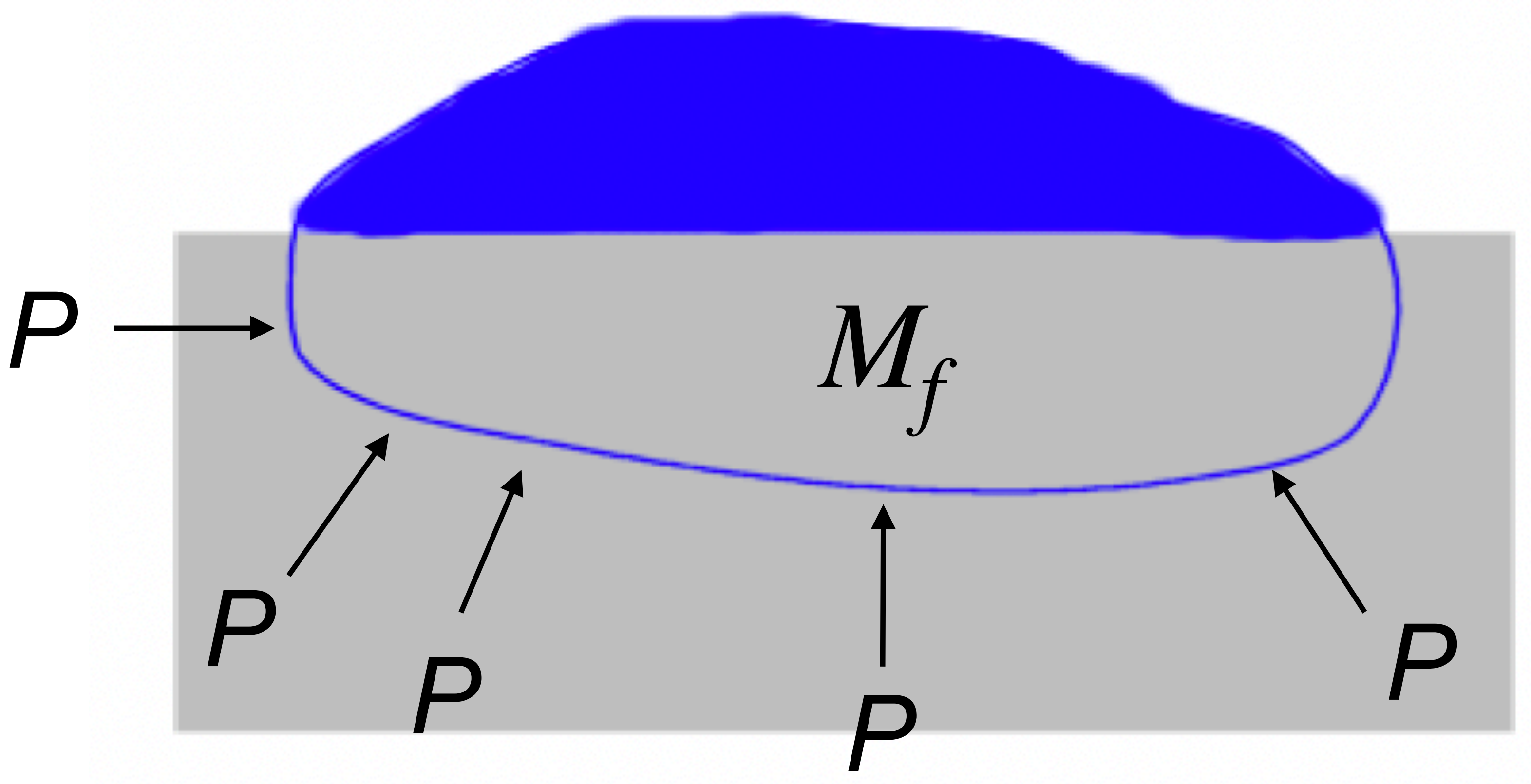


Imagine removing the body and replacing it by fluid.

Pressure $P(z)$ and density $\rho(z)$ remain the same.

Hydrostatic eq: $\vec{\nabla} P = \rho \vec{g}$

$$\int_V \vec{\nabla} P dV = \int \rho \vec{g} dV \quad \Rightarrow \quad \int_{\text{surface}} P d\vec{A} = M_f \vec{g}$$



M_f: mass of the fluid displaced by the object.

Archimedes' principle: $\vec{F}_{\text{buoy}} = -M_f \vec{g}$ (buoyant force = weight of fluid displaced by the object)

Tip of the Iceberg

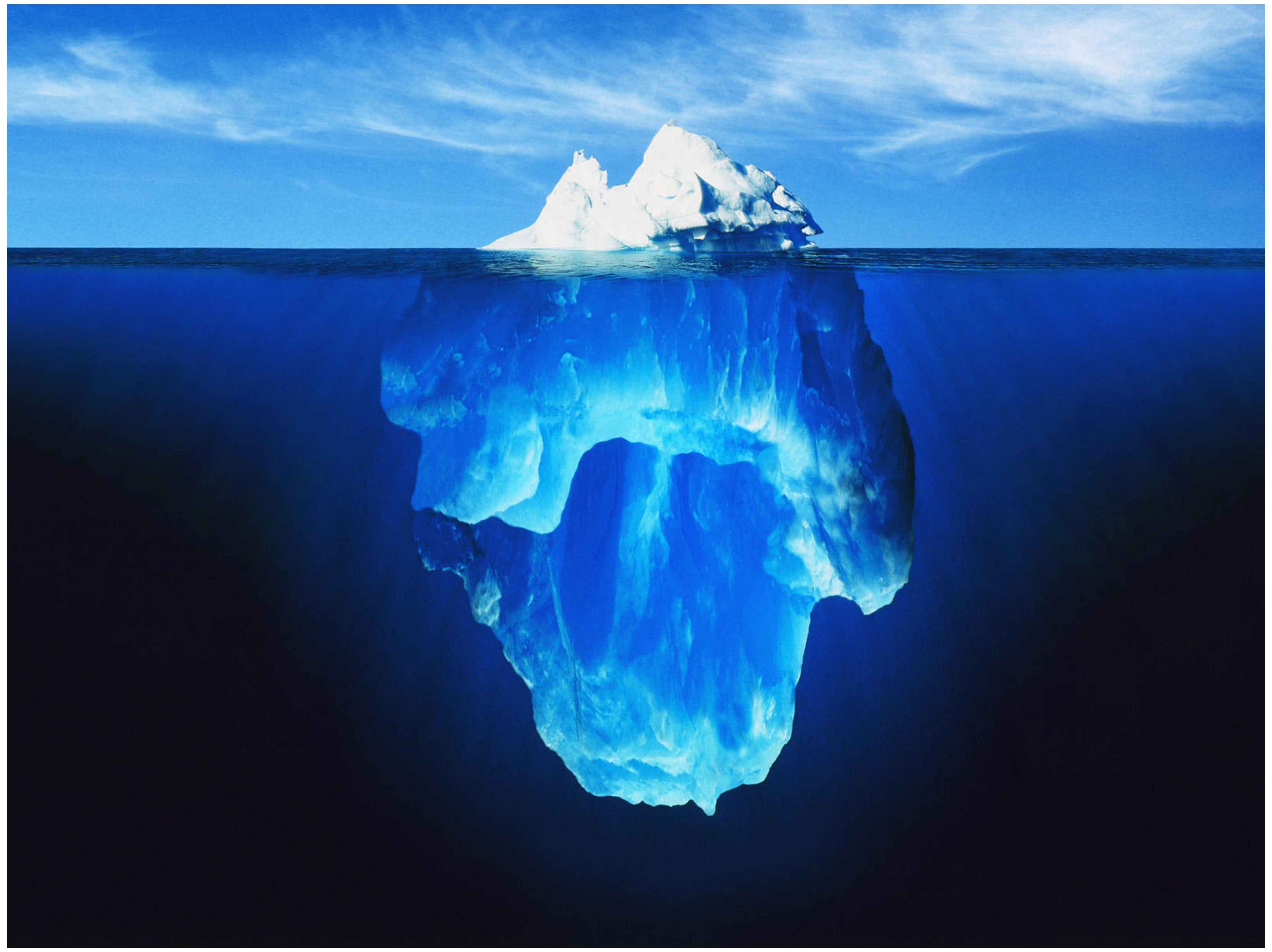
Density of ice $\rho_i = 920$ kg/m³
Density of sea water $\rho_w = 1027$ kg/m³
V_a : volume of iceberg above water
V : total volume of iceberg

In static state, weight of iceberg = buoyant force

$$\rho_i V g = \rho_w (V - V_a) g$$

$$\frac{V_a}{V} = \frac{\rho_w - \rho_i}{\rho_w} = 0.10$$

Only 10% of the iceberg is above the sea water!



Credit: clipground.com

Earth's Atmosphere I

Earth's atmospheric pressure is closely approximated by the hydrostatic equilibrium.

Let $\vec{g} = -g\hat{z}$ ($\hat{z}$ points upward).

$$\frac{dP}{dz} = -\rho g$$

ideal gas law: $P = nkT = \frac{\rho}{M}RT$

$R = N_A k =$ 8.31J/(mol K) = gas constant

M: molar mass of air = 0.02896 kg/mol (78% N_2, 21% O_2, 0.9% Ar and small amount of other gases)

$$\frac{dP}{dz} = -\frac{Mg}{RT}P \quad \Rightarrow \quad \frac{dP}{P} = -\frac{Mg}{RT}dz$$

$$P(z) = P_0 \exp\left(-\int_0^z \frac{Mg}{RT(z')}dz'\right)$$

P_0: pressure at z=0.

Earth's Atmosphere II

* If $T = T_0$ = constant (isothermal)

$$P(z) = P_0 e^{-Mgz/RT_0} \qquad \text{(isothermal)}$$

* If $T = T_0 - Lz$ (L is called the temperature lapse rate):

$$P(z) = P_0 \left(1 - \frac{Lz}{T_0}\right)^{Mg/RL} \qquad \text{(lapse)}$$

Recall:

$$\lim_{k\to\infty} \left(1 + \frac{x}{k}\right)^k = \lim_{k\to\infty} \exp\left[k \ln\left(1 + \frac{x}{k}\right)\right] = \lim_{k\to\infty} \exp\left(k \cdot \frac{x}{k}\right) = e^x$$

The lapse equation reduces to the isothermal equation in the limit $L \to 0$.

Earth's Atmosphere III

More realistic atmospheric model divides the atmosphere into several layers. Each layer has its own temperature lapse rate:

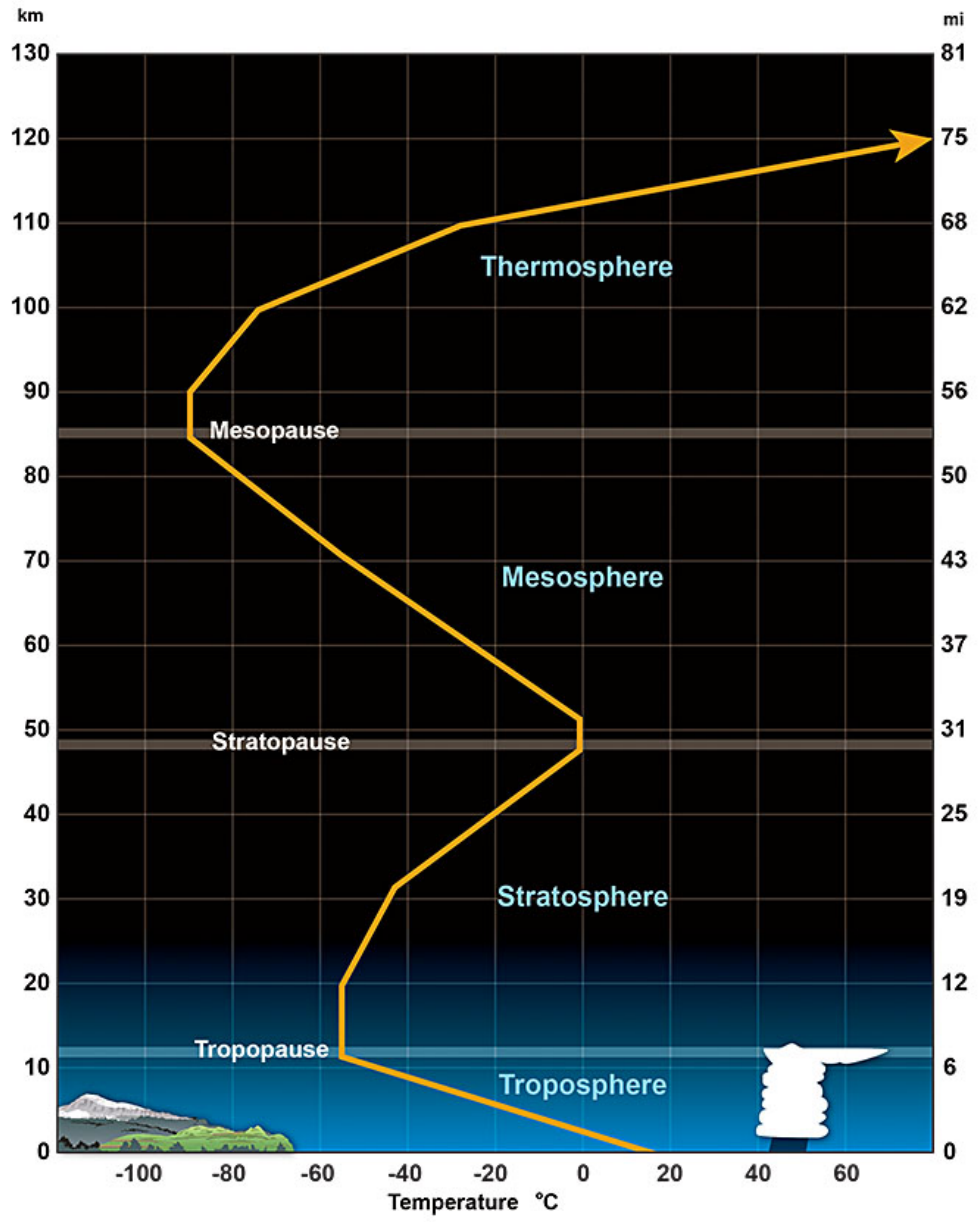
$$P(z) = P_b \left[1 - \frac{L_b(z - z_b)}{T_b}\right]^{Mg/RL_b}$$

P_b : pressure at the bottom of layer *b*.

T_b : temperature at the bottom of layer *b*.

L_b : temperature lapse rate in layer *b*.

z_b : altitude at the bottom of layer *b*.



Credit: NOAA

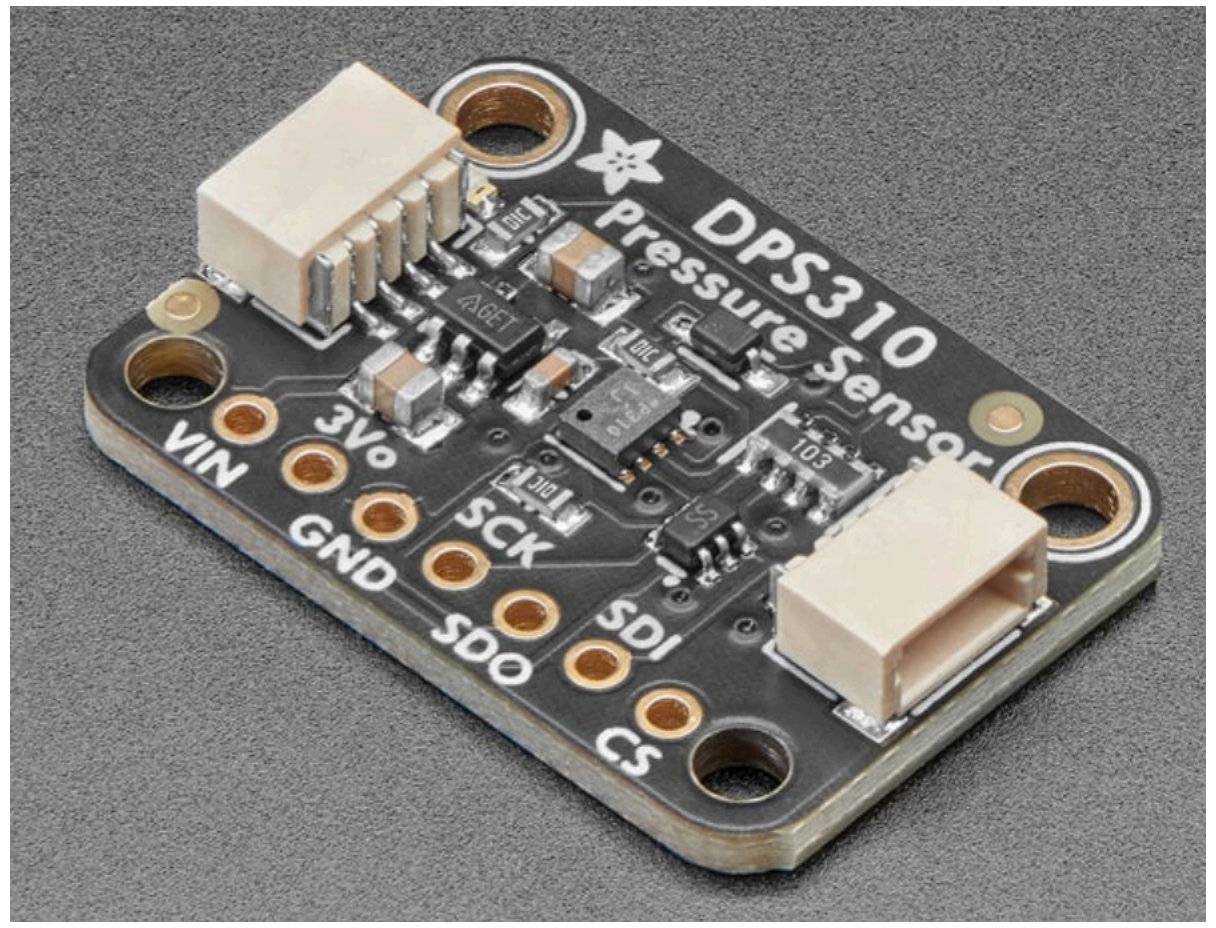
Earth's Atmosphere IV

Subscript b	Geopotential height above mean Sea level (z) z_b		Static pressure P_b		Standard temperature (K) T_b	Temperature lapse rate L_b	
	(m)	(ft)	(Pa)	(inHg)		(K/m)	(K/ft)
0	0	0	101 325.00	29.92126	288.15	0.0065	0.0019812
1	11 000	36,089	22 632.10	6.683245	216.65	0.0	0.0
2	20 000	65,617	5474.89	1.616734	216.65	-0.001	-0.0003048
3	32 000	104,987	868.02	0.2563258	228.65	-0.0028	-0.00085344
4	47 000	154,199	110.91	0.0327506	270.65	0.0	0.0
5	51 000	167,323	66.94	0.01976704	270.65	0.0028	0.00085344
6	71 000	232,940	3.96	0.00116833	214.65	0.002	0.0006096

Credit: Wikimedia (https://en.wikipedia.org/wiki/Barometric_formula)

DPS 310 Pressure Sensor

According to Adafruit, their DPS 310 pressure sensor can measure the change in pressure to an accuracy of 0.2 Pa.



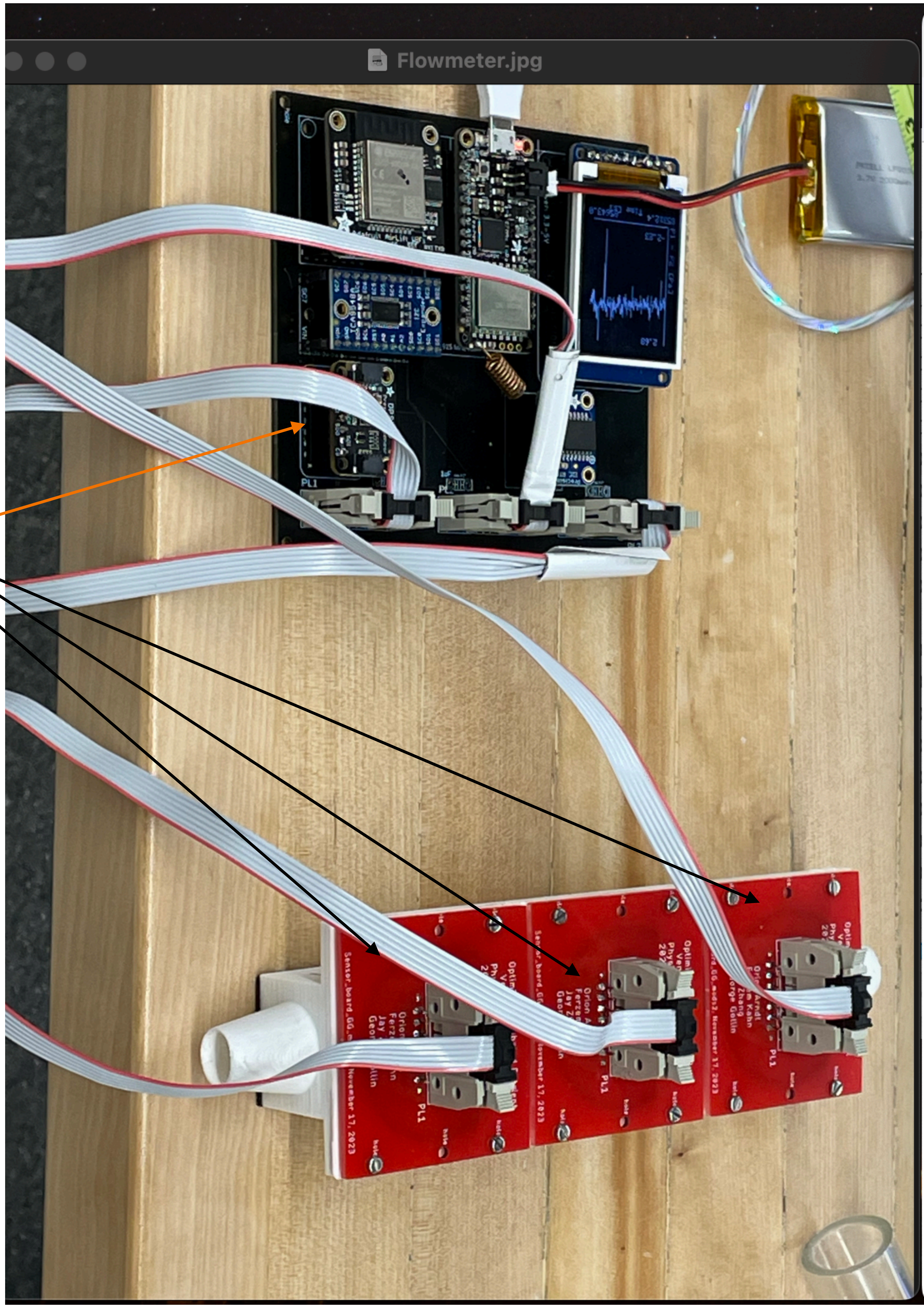
Credit: Adafruit

$$\frac{dP}{dz} = -\frac{Mg}{RT}P \qquad \Rightarrow \qquad \Delta P = -\frac{MgP}{RT}\Delta z$$

$\Delta P = 0.2$ Pa corresponds to $\Delta z = 1.7$ cm for $M = 0.02896$ kg/mol, P = 101 kPa, and T = 300 K.

Class Demonstration

4 DPS 310 sensors: 1 on the home board, 3 inside the flow tube



PHYS 525 :: Physics Illinois :: U

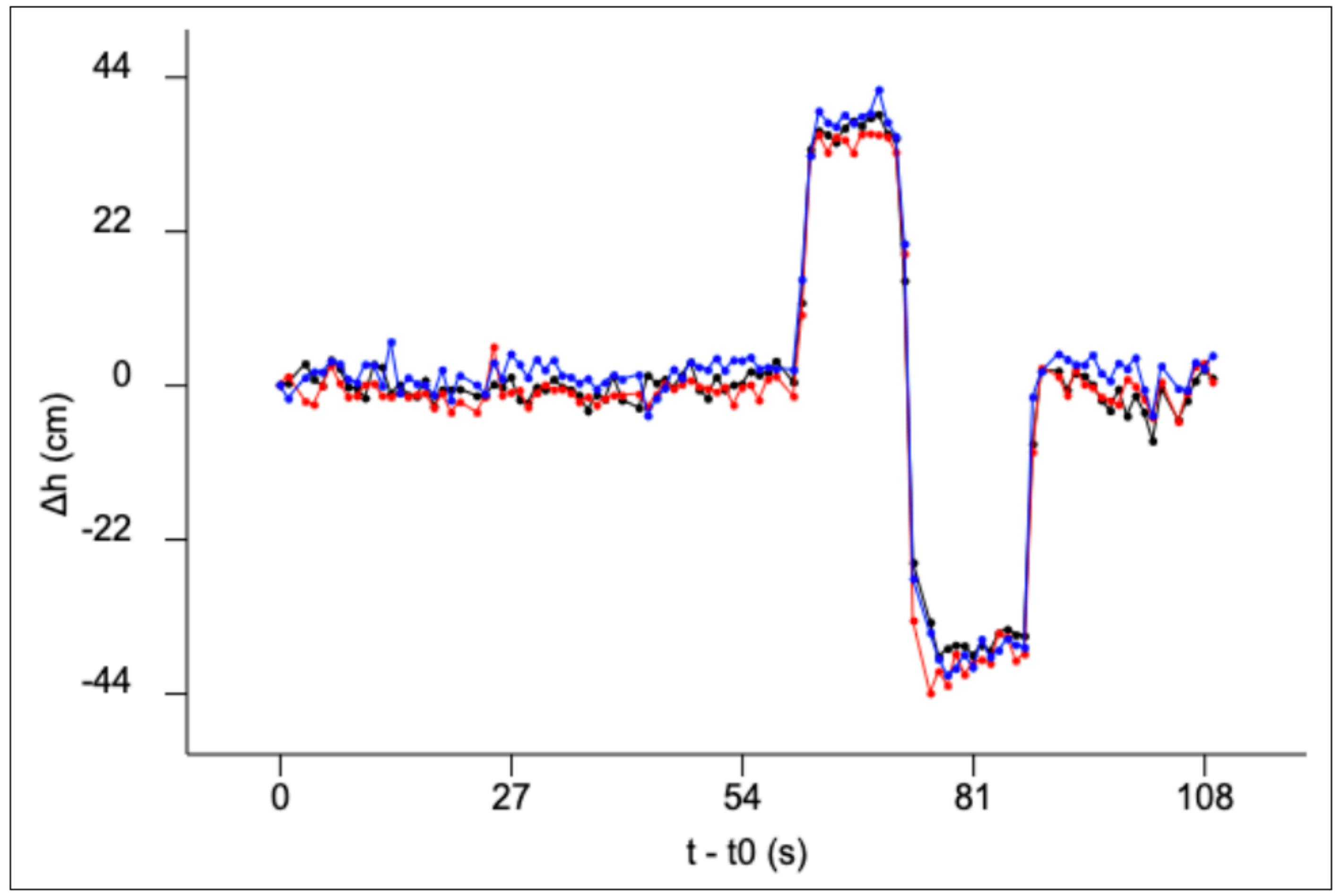
localhost/altitude.html

localhost/altitude.html

80%

Using the Ventilator Flowmeter as an Altitude meter

○ Simulated data ◉ Real data



t0: Sun Jan 02 2000 00:47:27 GMT-0600 (Central Standard Time)
Black: h1, Red: h2, Blue: h3

Number of points: 103

Latest time: Sun Jan 02 2000 00:49:16 GMT-0600 (Central Standard Time)

h1: 1 cm
h2: 0 cm
h3: 4 cm

Pressures recorded by the 4 sensors (Pa):
99816
99816.25
99815.97
99815.88

T0: 22.89°C
T1: 22.96°C
T2: 22.30°C
T3: 22.94°C

Gather the *P* and *T* readings from the 4 sensors and use them to calculate the altitudes of the 3 sensors in the tube relative to the sensor on the home board.

Energy Equation

Momentum equation: $\frac{d\vec{v}}{dt} = -\frac{\vec{\nabla} P}{\rho} + \vec{g}$

$\vec{v} \cdot \frac{d\vec{v}}{dt} = -\frac{\vec{v} \cdot \vec{\nabla} P}{\rho} + \vec{v} \cdot \vec{g}$, $\vec{v} \cdot \frac{d\vec{v}}{dt} = \frac{1}{2}\frac{d}{dt}(\vec{v} \cdot \vec{v}) = \frac{d}{dt}\left(\frac{v^2}{2}\right)$

$\vec{g} = -\vec{\nabla} U$, $U = gh$ is gravitational potential, *h* is height from a reference point.

Gravity is static near Earth's surface, $\partial U/\partial t = 0$.

$$\frac{dU}{dt} = \frac{\partial U}{\partial t} + \vec{v} \cdot \vec{\nabla} U = \vec{v} \cdot \vec{\nabla} U = -\vec{v} \cdot \vec{g}$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{1}{2}v^2 + U\right) + \frac{\vec{v} \cdot \vec{\nabla} P}{\rho} = 0$$

First Law of Thermodynamics

Consider a fluid element in a small volume *V*.

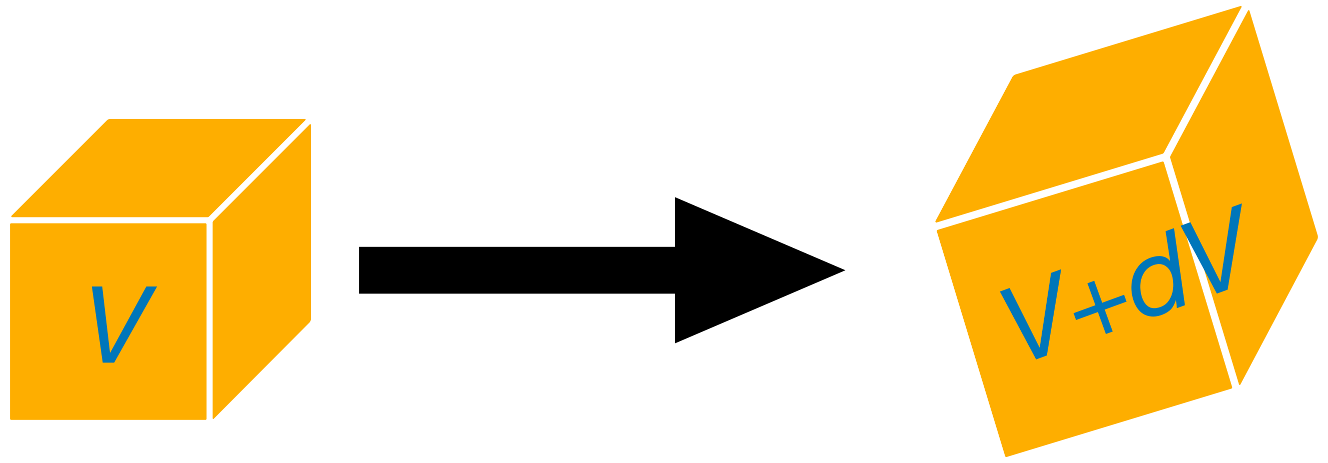
Mass $m = \rho V$, internal energy is *E*. First law of thermodynamics: $dE = dQ - PdV$

dQ is the amount of heat added to the volume. In the absence of heat generation and heat flow, dQ=0. The system is said to be adiabatic and $\frac{dE}{dt} = -P\frac{dV}{dt}$. Divide the equation by the mass $m = \rho V$ and write $w = E/m$ (specific internal energy).

$$\frac{dw}{dt} = -\frac{P}{\rho V}\frac{dV}{dt} = -P\frac{d}{dt}\left(\frac{V}{\rho V}\right) = -P\frac{d}{dt}\left(\frac{1}{\rho}\right) = -\frac{d}{dt}\left(\frac{P}{\rho}\right) + \frac{1}{\rho}\frac{dP}{dt}$$

$$\frac{d}{dt}\left(w + \frac{P}{\rho}\right) = \frac{1}{\rho}\frac{dP}{dt} = \frac{1}{\rho}\frac{\partial P}{\partial t} + \frac{\vec{v}\cdot\vec{\nabla}P}{\rho}$$

$$\frac{\vec{v}\cdot\vec{\nabla}P}{\rho} = \frac{d}{dt}\left(w + \frac{P}{\rho}\right) - \frac{1}{\rho}\frac{\partial P}{\partial t}$$



Volume moves with the fluid element

$m = \rho V = (\rho + d\rho)(V + dV)$

Bernoulli's Equation

Previous slides:

$$\frac{d}{dt}\left(\frac{1}{2}v^2 + U\right) + \frac{\vec{v}\cdot\overrightarrow{\nabla}P}{\rho} = 0 \quad , \quad \frac{\vec{v}\cdot\overrightarrow{\nabla}P}{\rho} = \frac{d}{dt}\left(w + \frac{P}{\rho}\right) - \frac{1}{\rho}\frac{\partial P}{\partial t}$$

Combine these two equations:

$$\frac{d}{dt}\left(\frac{1}{2}v^2 + \frac{P}{\rho} + U + w\right) = \frac{1}{\rho}\frac{\partial P}{\partial t}$$

In steady flow, $\partial P/\partial t = 0$, the resulting equation is called Bernoulli's equation.

$$\boxed{\frac{d}{dt}\left(\frac{1}{2}v^2 + \frac{P}{\rho} + U + w\right) = 0}$$

Recall: $\frac{dw}{dt} = -P\frac{d}{dt}\left(\frac{1}{\rho}\right) = 0$ for incompressible fluid $\Rightarrow$ $\frac{d}{dt}\left(\frac{1}{2}v^2 + \frac{P}{\rho} + U\right) = 0$

Bernoulli's Equation and Streamline

Let $b = \frac{1}{2}v^2 + \frac{P}{\rho} + gh + w$

Bernoulli's equation $\Rightarrow$ $\frac{db}{dt} = 0 \Rightarrow b =$ constant along a *streamline*

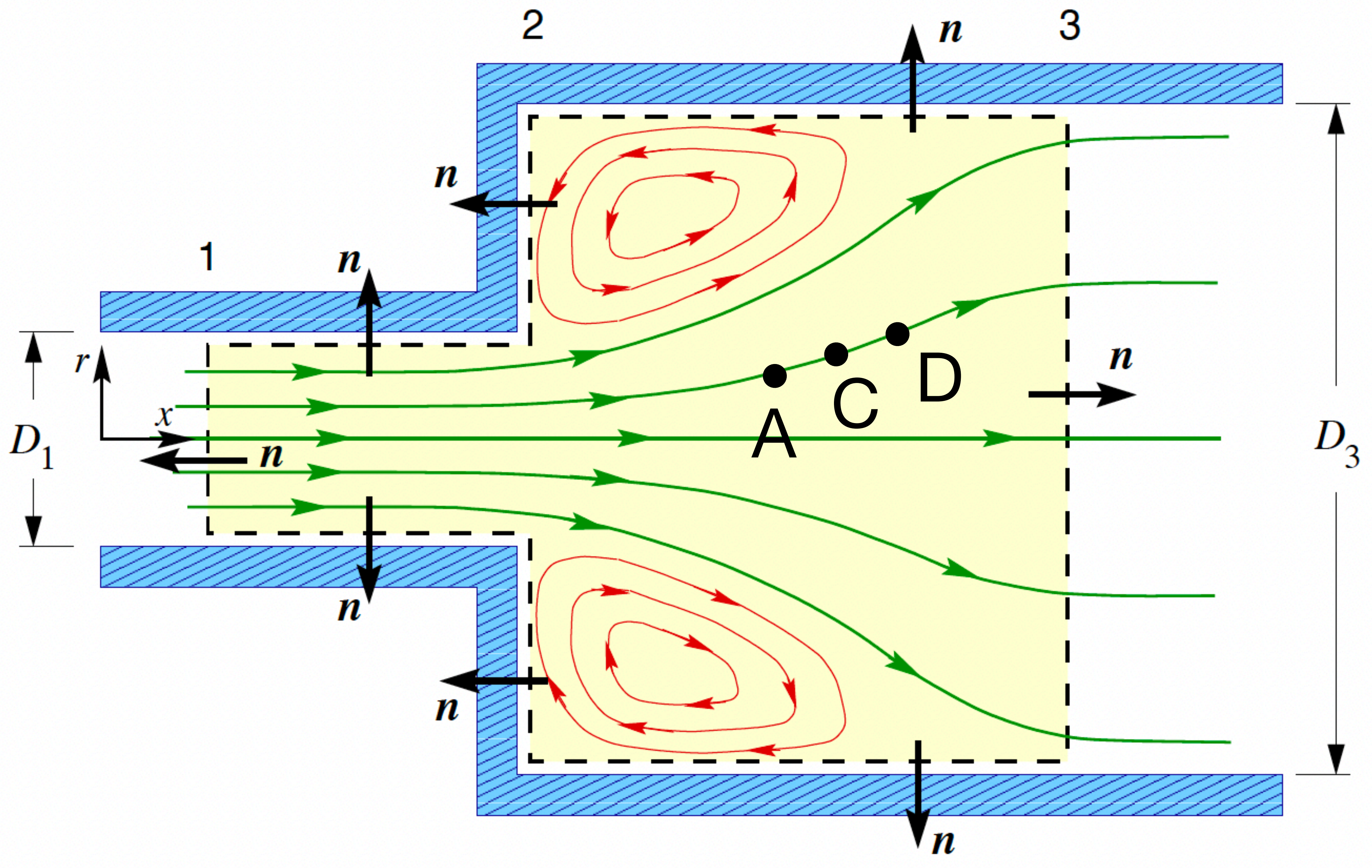
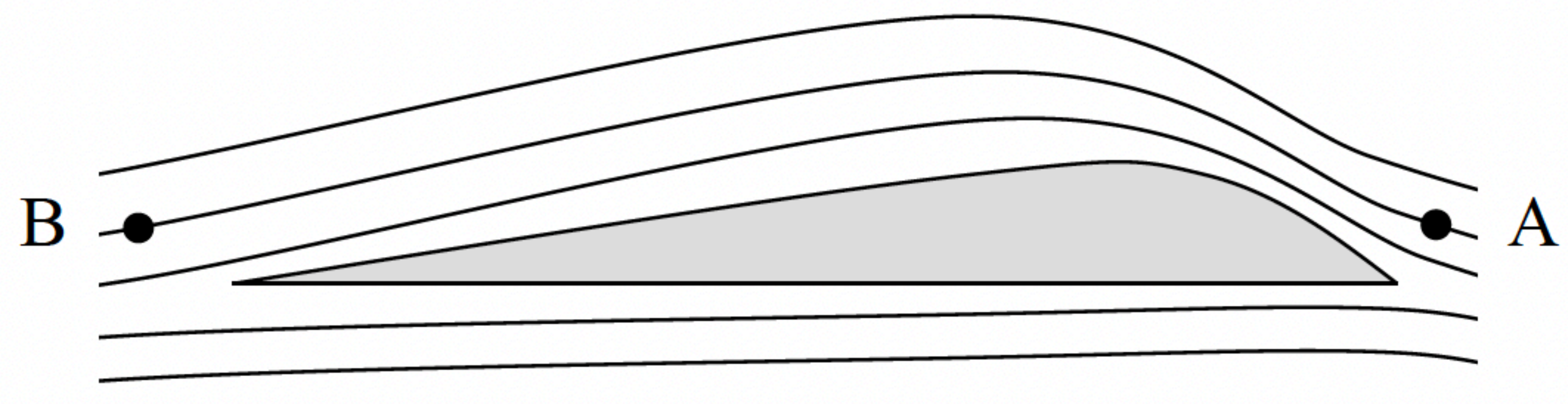


Figure 4.8: Flow through a rapidly-expanding pipe.

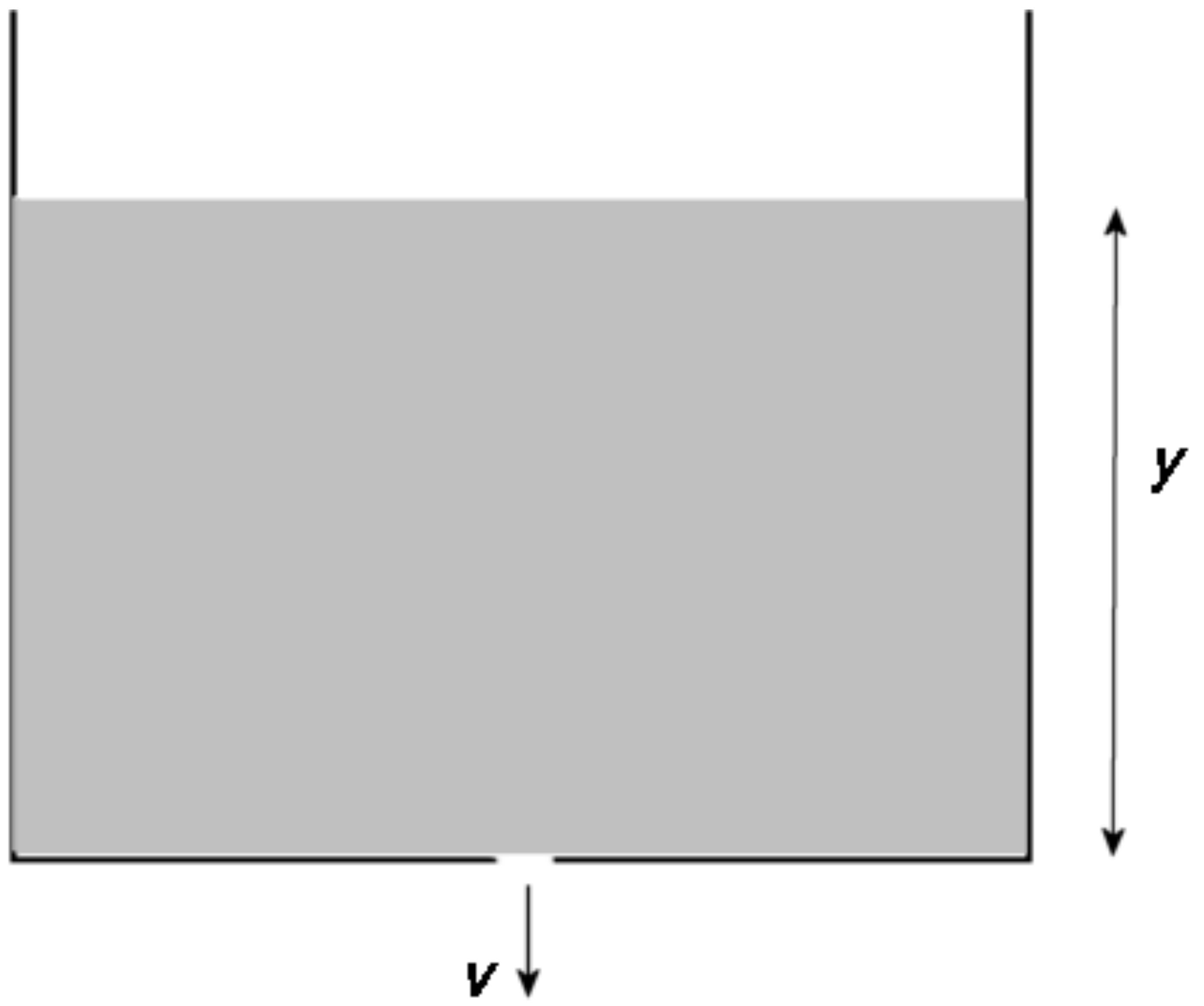


Bernoulli's equation doesn't apply to turbulent flows.
* Turbulent flows are usually not steady
* No well-defined streamlines
* Viscosity is important

Figure credit: J.M. McDonough, Lectures In Elementary Fluid Dynamics: Physics, Mathematics and Applications

Example

Water is flowing out of a rectangular tank from a small hole at the bottom. How long does it take to excavate the water from the tank?



Apply Bernoulli's equation at the top and at the hole:

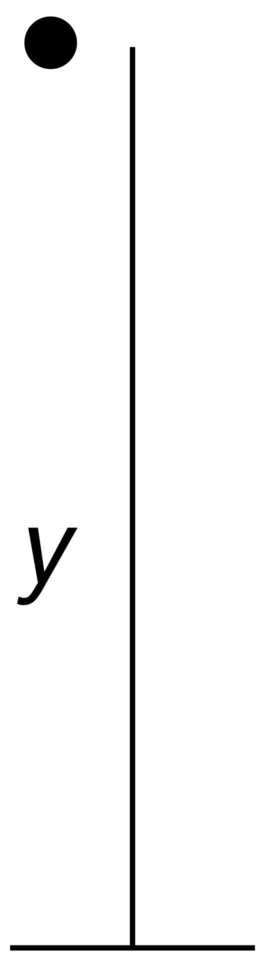
$$\frac{1}{2}\dot{y}^2 + \frac{P}{\rho} + gy = \frac{1}{2}v^2 + \frac{P}{\rho} \quad \Rightarrow \quad v^2 - \dot{y}^2 = 2gy$$

Previously, we find $\dot{y} = -\frac{A_h}{A}v$

A_h : area of the hole, *A*: cross-sectional area of the tank.

$$\Rightarrow \left(1 - \frac{A_h^2}{A^2}\right)v^2 = 2gy \quad ,$$

$$v = \sqrt{2gy}\left(1 - \frac{A_h^2}{A^2}\right)^{-1/2} \approx \sqrt{2gy} \quad \text{for } A_h \ll A$$



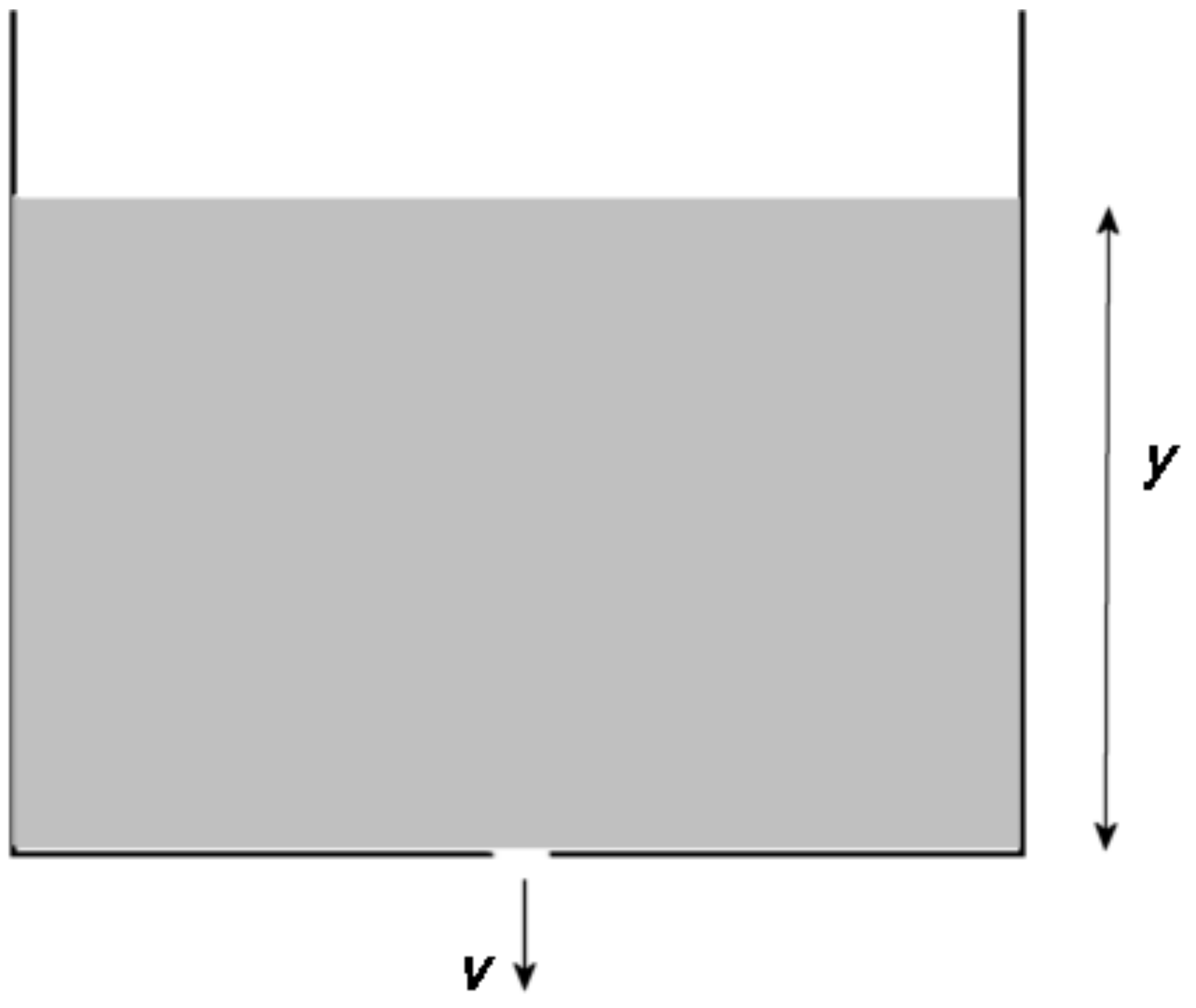
Free fall from *y*:

$$\frac{1}{2}mv^2 = mgy$$

$$\Rightarrow \quad v = \sqrt{2gy}$$

This is the free-fall speed from *y*. As the water level drops, the speed also decreases.

Example (cont)



Rate of change of water level: $\dot{y} = -\frac{A_h}{A} v = -\frac{A_h}{A}\sqrt{2gy}$

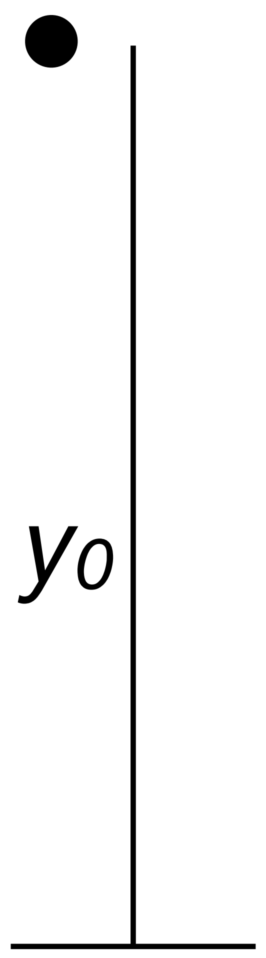
$$\frac{dy}{\sqrt{y}} = -\frac{A_h}{A}\sqrt{2g}\,dt$$

Let $y_0 = y(t=0)$. Integrate both sides:

$$\int_{y_0}^{y} \frac{dy'}{\sqrt{y'}} = -\frac{A_h}{A}\sqrt{2g}\ t \quad , \quad 2\sqrt{y} - 2\sqrt{y_0} = -\frac{A_h}{A}\sqrt{2g}\ t$$

$$y(t) = \left(\sqrt{y_0} - \frac{A_h}{A}\sqrt{\frac{g}{2}}\,t\right)^2$$

Setting $y(T) = 0$ gives $T = \frac{A}{A_h}\sqrt{\frac{2y_0}{g}} = \frac{A}{A_h} \times$ free-fall time.



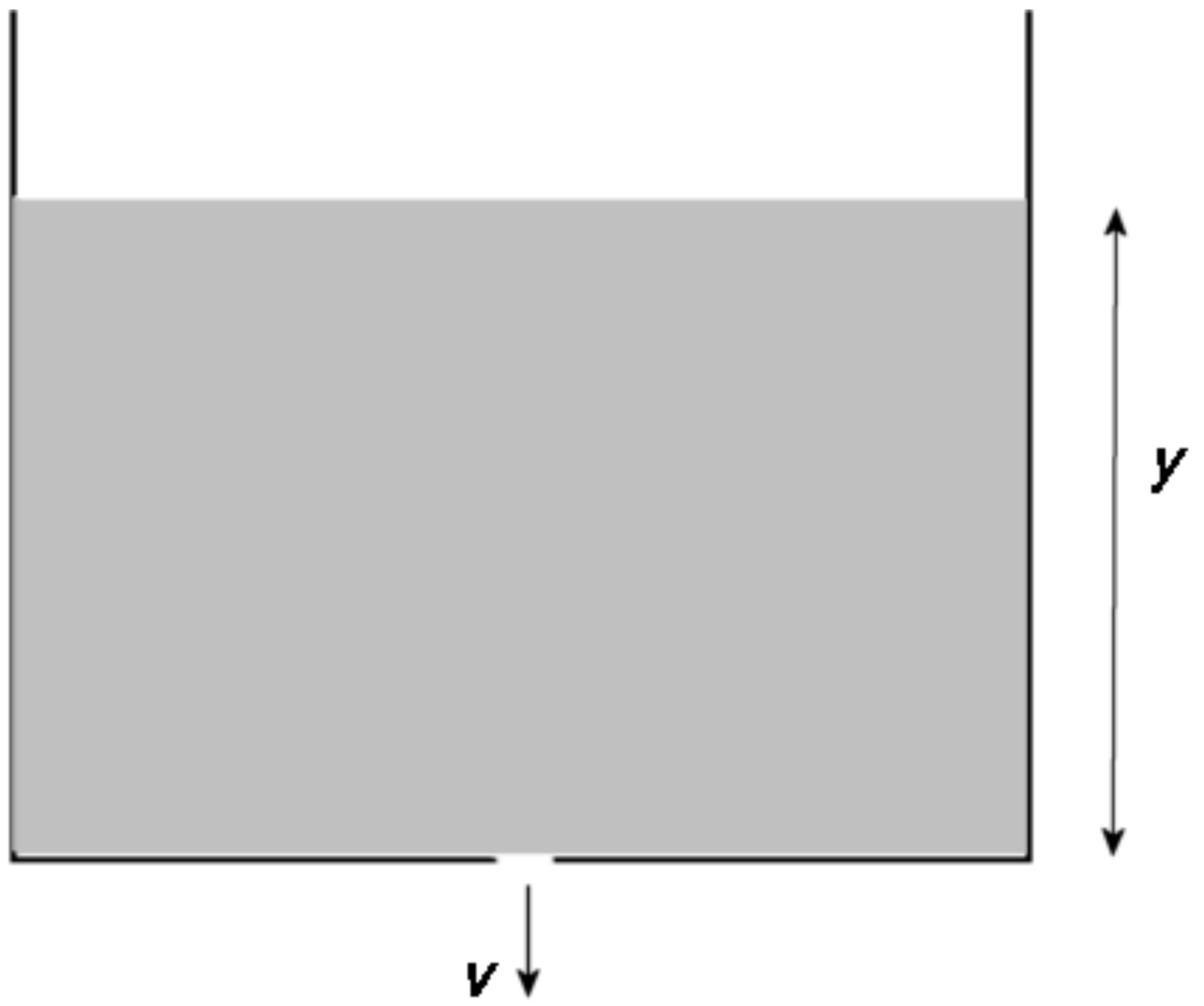
Free fall from y_0:

$s = \frac{1}{2}gt^2$

$s = y_0$ when

$t = \sqrt{2y_0/g}$

Example (cont)



$$T = \frac{A}{A_h}\sqrt{\frac{2y_0}{g}}$$

For y_0=0.3 m, $A/A_h = 40$, $T \approx 10$ s.

Bernoulli's equation only applies to steady flow.

It's still a good approximation if the rate of change is sufficiently slow, which requires $T \gg$ dynamical time scales.

Two dynamical time scales:

(1) Time associated with pressure ~ time for sound to travel y_0 :
$\tau = y_0/c_s$. Sound speed in water ≈ 1500 m/s, $\tau \approx 0.0002$ s $\ll T$.

(2) Time associated with gravity ~ free-fall time.
$T = A/A_h \times$ free-fall time = 40 free-fall time.
Relative error in estimated $T \sim 1/40 = 2.5\,\%$.

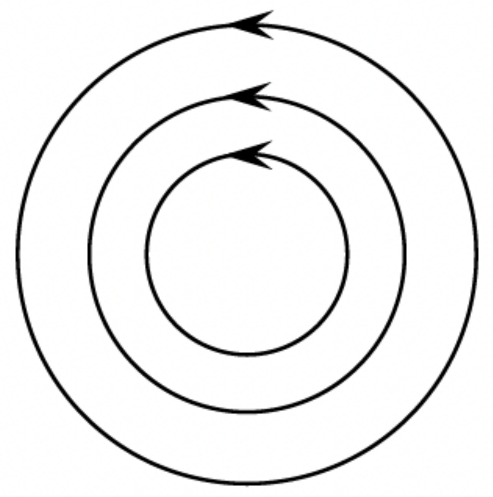
Vorticity

Vorticity is defined as $\overrightarrow{\omega} = \overrightarrow{\nabla} \times \vec{v}$. In Cartesian coordinates,

$$\overrightarrow{\omega} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{x} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{y} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{z}$$

It describes the local spinning motion of fluid.

Consider the velocity in the fluid near a vortex looks like this:



The velocity field is given by $\vec{v} = \overrightarrow{\Omega} \times \vec{r}$, where $\overrightarrow{\Omega}$ is a constant vector.

In cylindrical coordinates with $\overrightarrow{\Omega} = \Omega \hat{z}$, we have $v_\phi = \Omega r$ and $v_r = v_z = 0$.

$$\overrightarrow{\omega} = \frac{1}{r} \frac{\partial}{\partial r} (r v_\phi) \hat{z} = 2\Omega \hat{z}$$

The fluid is irrotational if $\overrightarrow{\omega} = 0$.

Vector Derivatives in Cylindrical Coordinates

CYLINDRICAL $d\mathbf{l} = dr\,\hat{r} + r\,d\phi\,\hat{\phi} + dz\,\hat{z}$; $d\tau = r\,dr\,d\phi\,dz$

Gradient. $$\nabla t = \frac{\partial t}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial t}{\partial \phi}\hat{\phi} + \frac{\partial t}{\partial z}\hat{z}$$

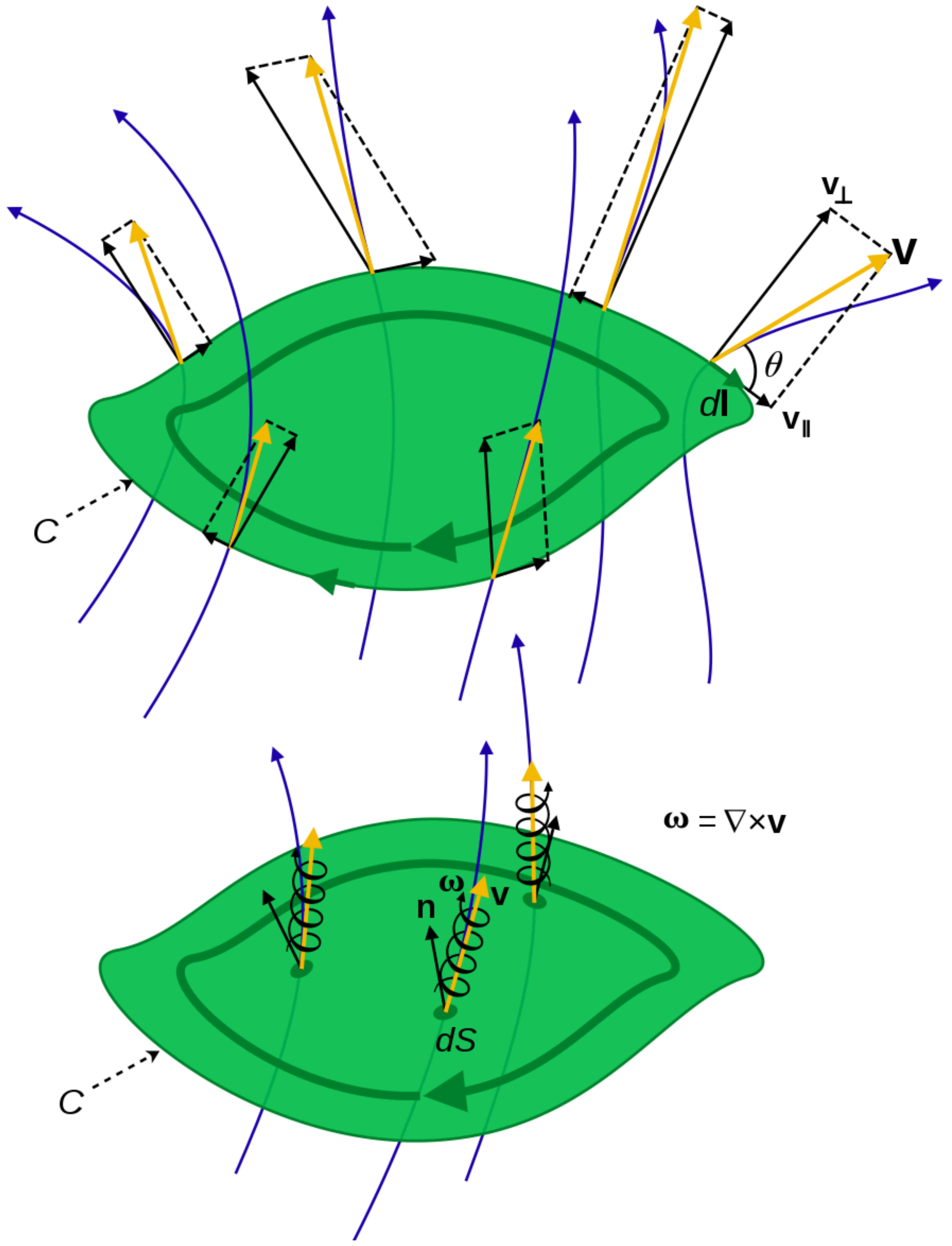
Divergence. $$\nabla\cdot\mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl. $$\nabla\times\mathbf{v} = \left[\frac{1}{r}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{r} + \left[\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right]\hat{\phi} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rv_\phi) - \frac{\partial v_r}{\partial \phi}\right]\hat{z}$$

Laplacian. $$\nabla^2 t = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial t}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$

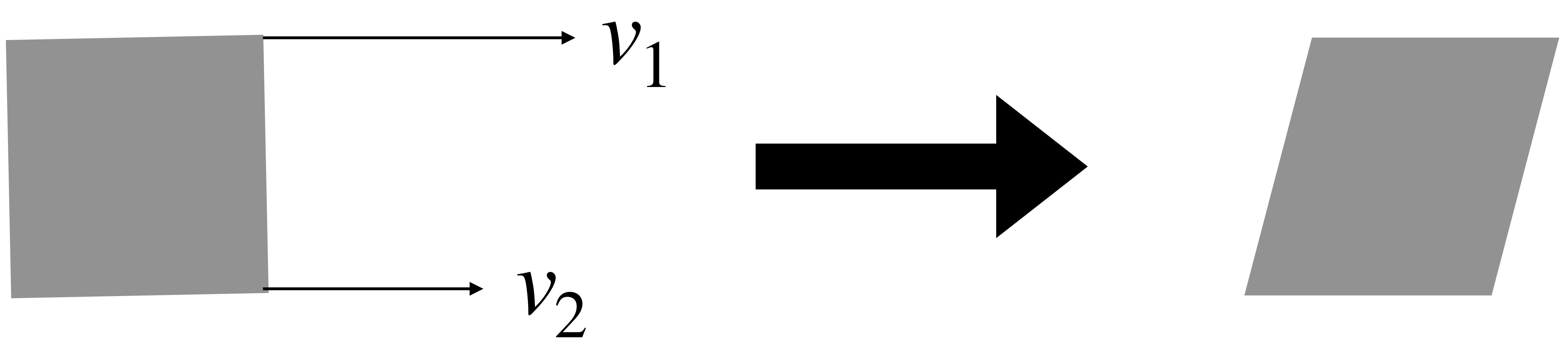
Circulation

- Circulation is closely related to vorticity
- Circulation of a fluid around a closed loop is defined as
$$\Gamma = \oint \vec{v} \cdot d\vec{l}$$
- Stoke's theorem:
$$\Gamma = \int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{S} = \int_S \vec{\omega} \cdot d\vec{S}$$
- If the flow is irrotational, $\vec{\omega} = 0 \Rightarrow \Gamma = 0.$



Credit: Wikipedia

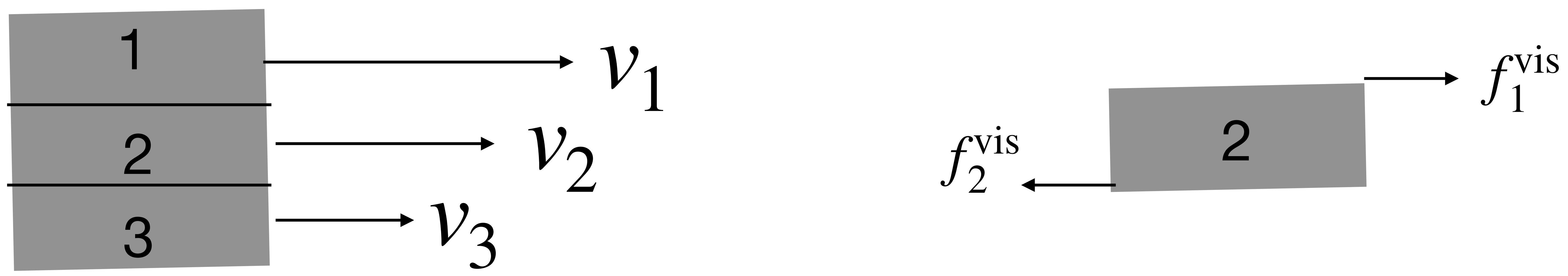
Shearing



Shearing can occur when neighboring fluid moves with different velocities.

In the presence of viscosity, the shear motion develops a viscous stress that opposes the motion.

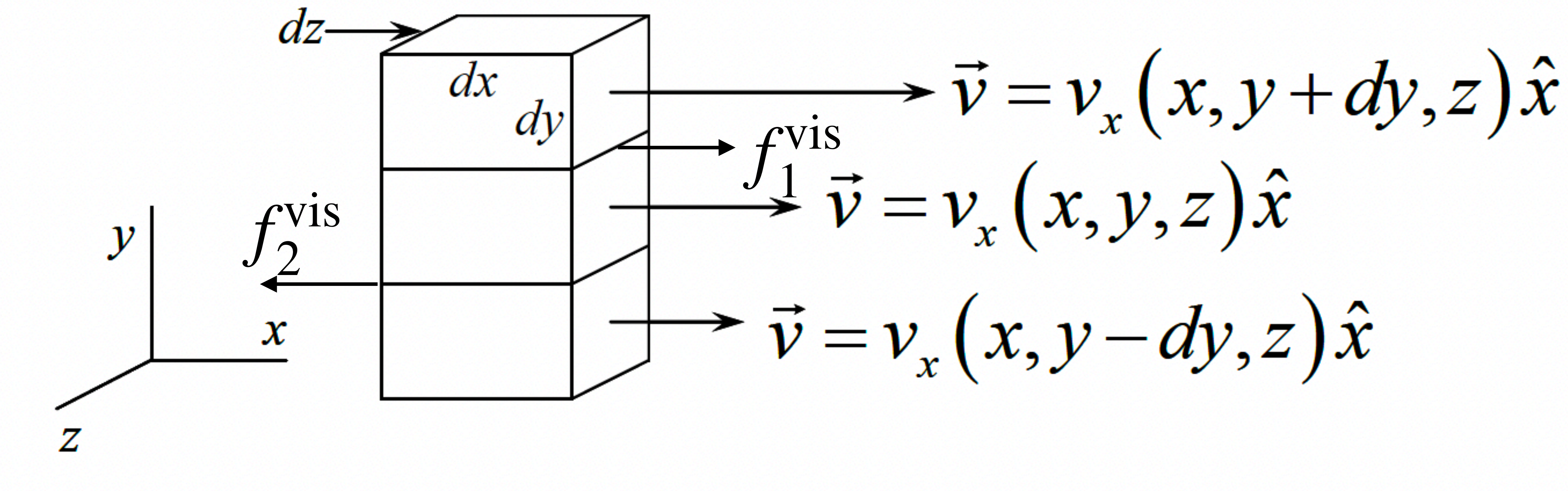
The stress acting on a fluid element can be characterized by a stress tensor $\overleftrightarrow{T}$.



Simple Model of Viscosity

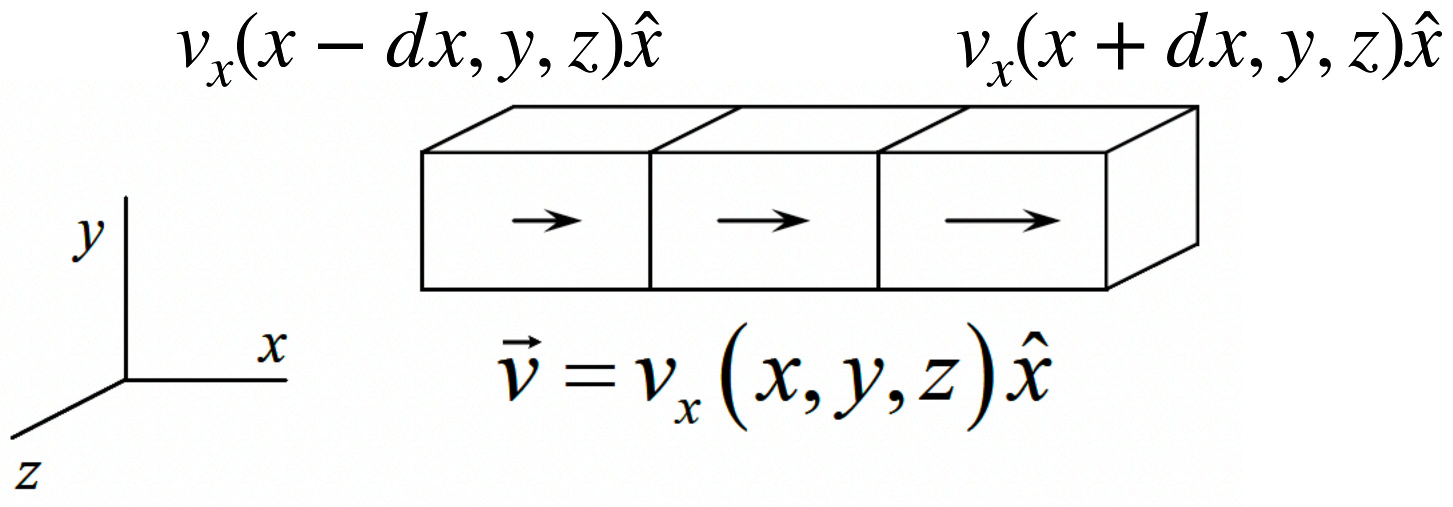
$$f_1^{\text{vis}} = \mu \frac{\partial v_x(x, y + dy/2, z)}{\partial y} dxdz$$

$$f_2^{\text{vis}} = -\mu \frac{\partial v_x(x, y - dy/2, z)}{\partial y} dxdz$$



μ : coefficient of shear viscosity

Net force $f_x^{\text{vis}} = f_1^{\text{vis}} + f_2^{\text{vis}} = \mu \frac{\partial^2 v_x}{\partial y^2} dxdydz = \mu \frac{\partial^2 v_x}{\partial y^2} dV$



Adding the contributions from the other two directions:

$$f_x^{\text{vis}} = \mu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) dV = \mu \nabla^2 v_x dV$$

The y and z-components of the viscous force are obtained by changing v_x to v_y and v_z.

Viscous force: $\vec{f}^{\text{vis}} = \mu \nabla^2 \vec{v} dV$

Stress Tensor

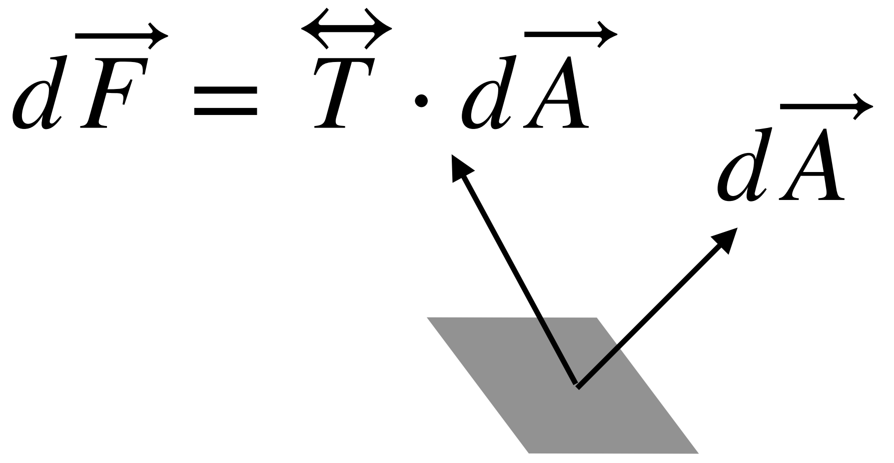
- Stress tensor can be represented by a 3×3 matrix. In Cartesian coordinates,

$$\overleftrightarrow{T} = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}$$

- Force acting on a small surface $d\overrightarrow{A} = \hat{n} dA$ is given by

$$d\overrightarrow{F} = \overleftrightarrow{T} \cdot d\overrightarrow{A} = dA(T_{xx}n_x + T_{xy}n_y + T_{xz}n_z)\hat{x} + dA(T_{yx}n_x + T_{yy}n_y + T_{yz}n_z)\hat{y} + dA(T_{zx}n_x + T_{zy}n_y + T_{zz}n_z)\hat{z}$$

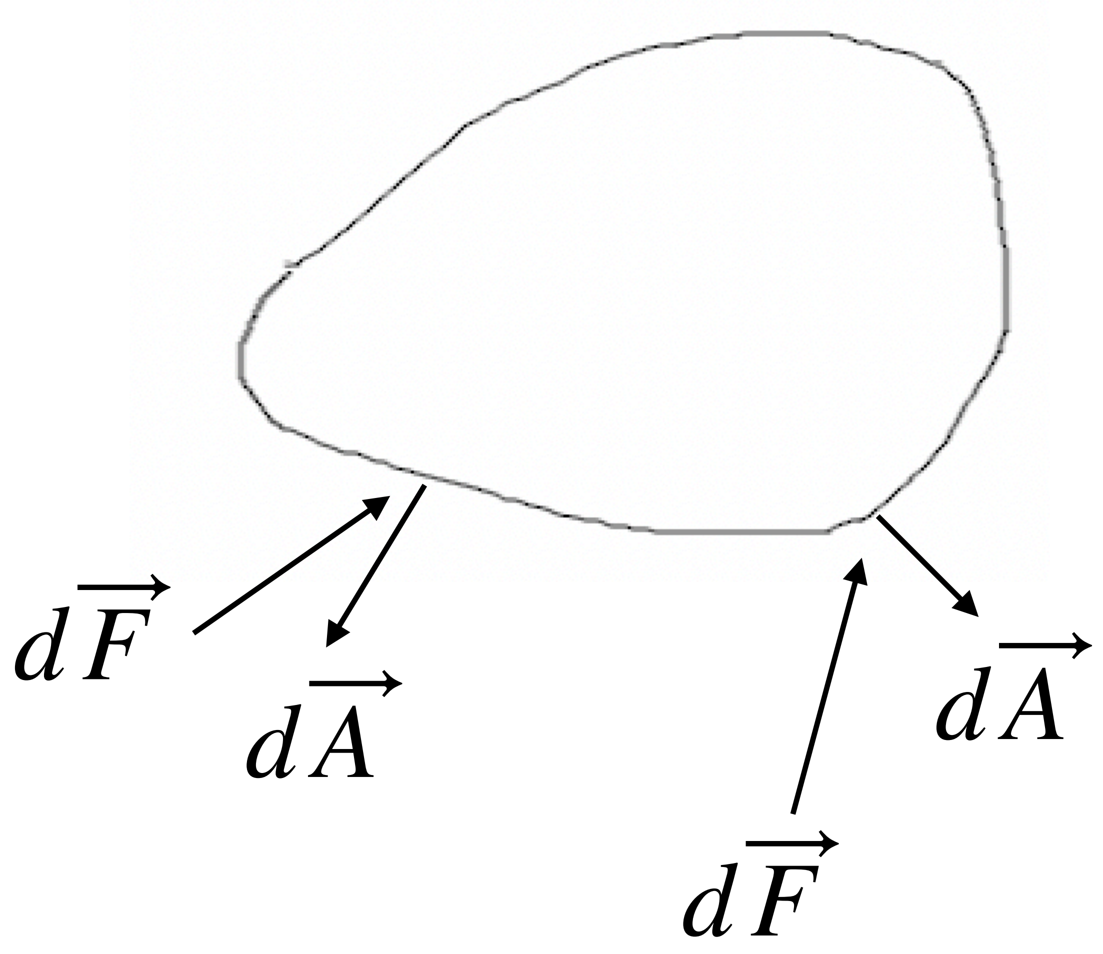
$$= dA \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix} \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}$$



- It can be shown that $\overleftrightarrow{T}$ must be symmetry: $T_{ij} = T_{ji}$

Force on Fluid

$$\overrightarrow{F} = -\int_S \overleftrightarrow{T} \cdot d\overrightarrow{A}$$



Note the negative sign since $d\overrightarrow{A}$ points outward.

Divergence theorem:

$$\overrightarrow{F} = -\int_V \overrightarrow{\nabla} \cdot \overleftrightarrow{T} dV$$

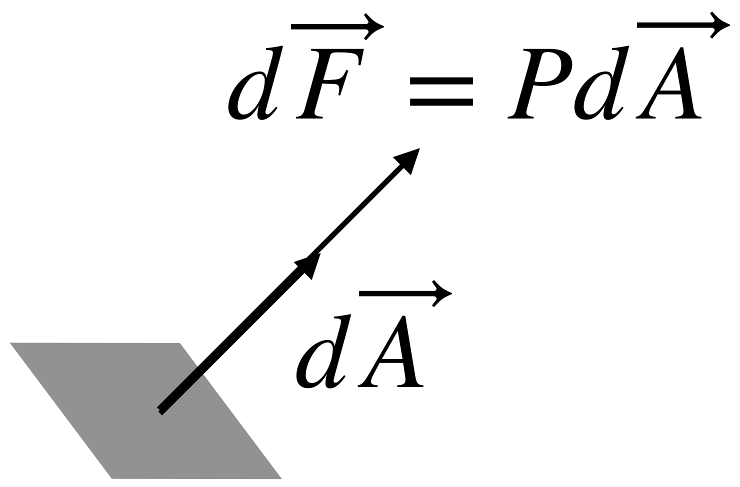
$$\overrightarrow{\nabla} \cdot \overleftrightarrow{T} = \left(\frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{yx}}{\partial y} + \frac{\partial T_{zx}}{\partial z} \right) \hat{x} + \left(\frac{\partial T_{xy}}{\partial x} + \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{zy}}{\partial z} \right) \hat{y} + \left(\frac{\partial T_{xz}}{\partial x} + \frac{\partial T_{yz}}{\partial y} + \frac{\partial T_{zz}}{\partial z} \right) \hat{z}$$

Force per unit volume: $\vec{f} = -\overrightarrow{\nabla} \cdot \overleftrightarrow{T}$

Viscous Stress Tensor

The stress tensor of an ideal fluid is $\overleftrightarrow{T} = P\overleftrightarrow{G}$, where $\overleftrightarrow{G}$ is called the metric tensor and is represented by an identity matrix in Cartesian coordinates. In Cartesian coordinates, $\overleftrightarrow{T}$ is represented by a diagonal matrix

$$\overleftrightarrow{T} = \begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & P \end{pmatrix}$$



Force acting on a small area is $d\overrightarrow{F} = \overleftrightarrow{T} \cdot d\overrightarrow{A} = Pd\overrightarrow{A}$. Force is isotropic (same magnitude in every direction). Force per unit volume is

$$\vec{f} = -\overrightarrow{\nabla} \cdot \overleftrightarrow{T} = -\frac{\partial P}{\partial x}\hat{x} - \frac{\partial P}{\partial y}\hat{y} - \frac{\partial P}{\partial z}\hat{z} = -\overrightarrow{\nabla} P$$

In the presence of viscosity, $\overleftrightarrow{T} = P\overleftrightarrow{G} + \overleftrightarrow{\tau}$, $\overleftrightarrow{\tau}$ is called the viscous stress tensor.

Viscous force acting on a small ares is $d\overrightarrow{F}_{\text{vis}} = \overleftrightarrow{\tau} \cdot d\overrightarrow{A}$

Viscous force per unit volume is $\vec{f}_{\text{vis}} = -\overrightarrow{\nabla} \cdot \overleftrightarrow{\tau}$

Momentum Equation with Viscosity

Momentum equation: $(\rho dV)\dfrac{d\vec{v}}{dt} = -\,dV\vec{\nabla}\cdot\overleftrightarrow{T} + (\rho dV)\vec{g}$

$$\rho\frac{d\vec{v}}{dt} = -\,\vec{\nabla}\cdot\overleftrightarrow{T} + \rho\vec{g}$$

$$\overleftrightarrow{T} = P\overleftrightarrow{G} + \overleftrightarrow{\tau} \qquad \Rightarrow \qquad \vec{\nabla}\cdot\overleftrightarrow{T} = \vec{\nabla}P + \vec{\nabla}\cdot\overleftrightarrow{\tau}$$

$$\frac{d\vec{v}}{dt} = \frac{\partial\vec{v}}{\partial t} + \vec{v}\cdot\vec{\nabla}\vec{v} = -\frac{\vec{\nabla}P}{\rho} + \vec{g} - \frac{1}{\rho}\vec{\nabla}\cdot\overleftrightarrow{\tau}$$

Need an expression for $\overleftrightarrow{\tau}$ that depends on the velocity field $\vec{v}$.

$\overleftrightarrow{\tau} \neq 0$ only for non-uniform $\vec{v}$, but $\overleftrightarrow{\tau} = 0$ if the fluid is rigidly rotating.

Velocity Gradient Tensor

The velocity gradient tensor $\vec{\nabla}\vec{v}$ can be represented by a matrix: $\vec{\nabla}\vec{v} = \begin{pmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_y}{\partial x} & \frac{\partial v_z}{\partial x} \\ \frac{\partial v_x}{\partial y} & \frac{\partial v_y}{\partial y} & \frac{\partial v_z}{\partial y} \\ \frac{\partial v_x}{\partial z} & \frac{\partial v_y}{\partial z} & \frac{\partial v_z}{\partial z} \end{pmatrix}$

$\overleftrightarrow{\tau}$ is symmetric, but $\vec{\nabla}\vec{v}$ is not. Cannot express $\overleftrightarrow{\tau}$ in terms of $\vec{\nabla}\vec{v}$ directly.

Decompose $\vec{\nabla}\vec{v}$ into 3 components: $(\vec{\nabla}\vec{v})_{ij} = \frac{\partial v_j}{\partial x_i} = \frac{1}{3}\theta\delta_{ij} + r_{ij} + \sigma_{ij}$

Expansion: $\theta = Tr(\vec{\nabla}\vec{v}) = \vec{\nabla} \cdot \vec{v}$

Anti-symmetric part of $\vec{\nabla}\vec{v}$: $r_{ij} = \frac{1}{2}\left(\frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j}\right)$

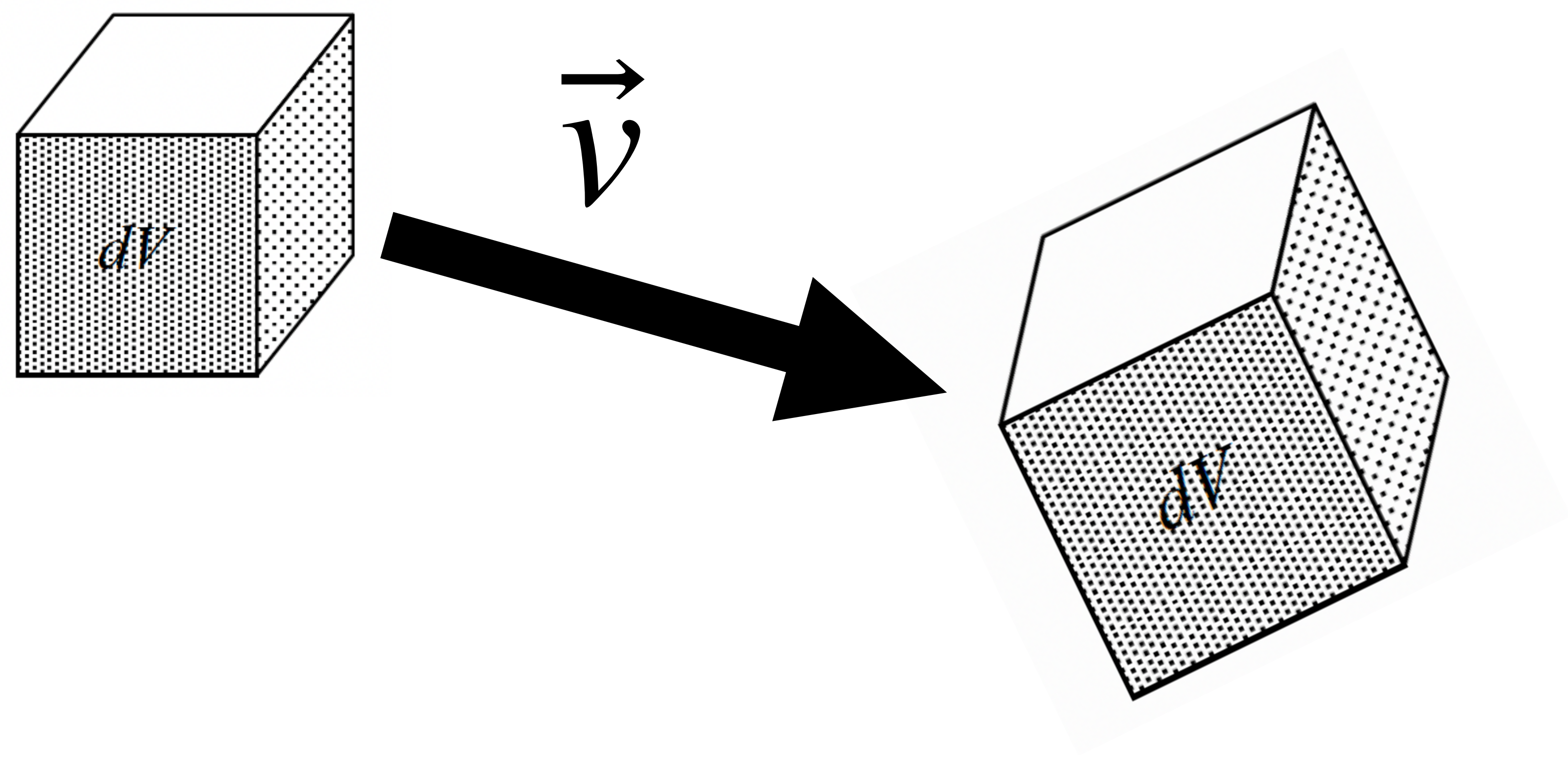
Symmetric trace-free part of $\vec{\nabla}\vec{v}$: $\sigma_{ij} = \frac{1}{2}\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right) - \frac{1}{3}\theta\delta_{ij}$

Physical Meaning of θ

Consider a small fluid element occupying a small volume ΔV and mass $\Delta m = \rho \Delta V$.

Moving with the mass, we have

$$0 = \frac{d}{dt}(\rho \Delta V) = \Delta V \frac{d\rho}{dt} + \rho \frac{d\Delta V}{dt}$$



Continuity equation: $\dfrac{d\rho}{dt} = -\rho \vec{\nabla} \cdot \vec{v} = -\rho\theta$

$$-\rho\theta\Delta V + \rho \frac{d\Delta V}{dt} = 0$$

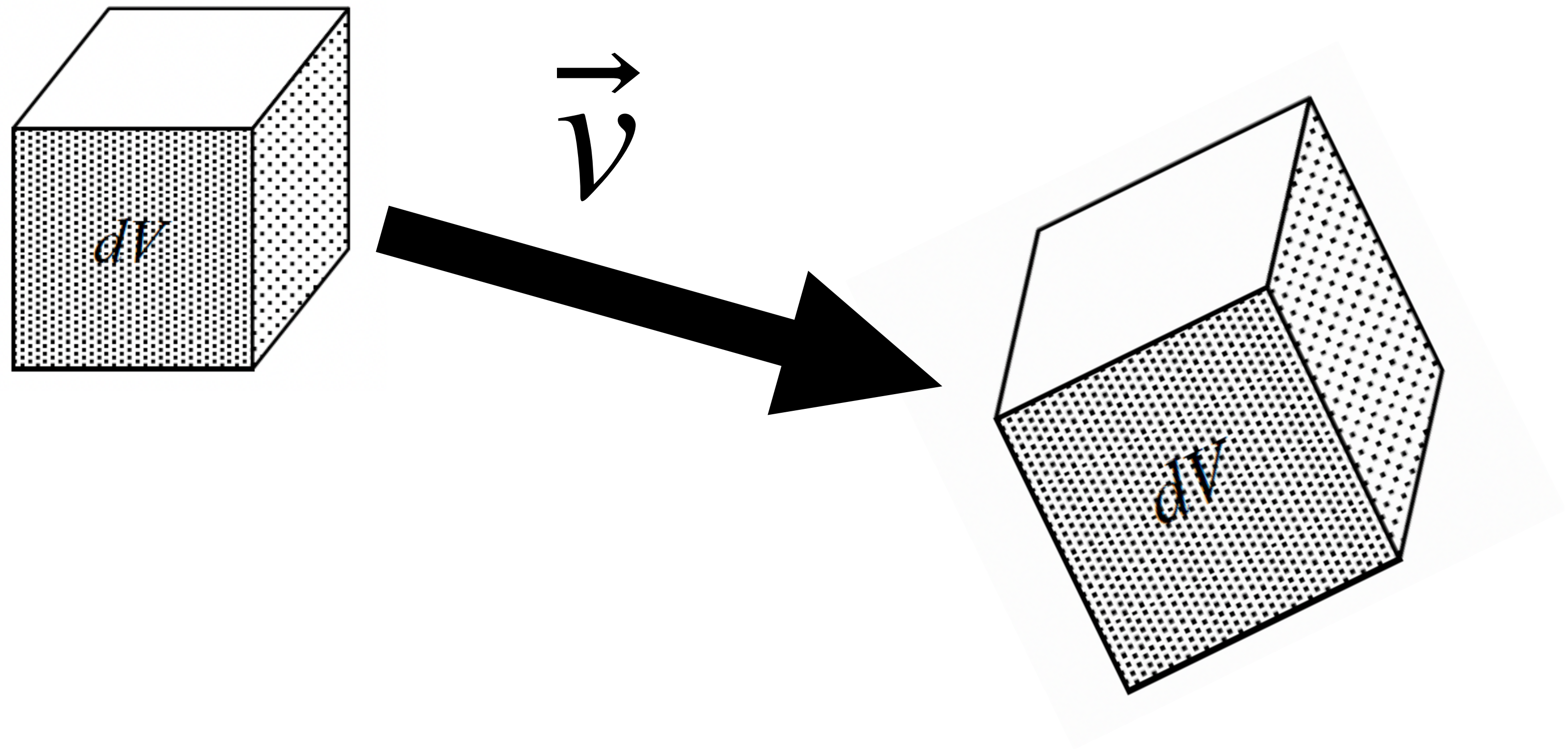
$$\theta = \frac{1}{\Delta V} \frac{d\Delta V}{dt}$$

θ is the fractional rate of increase of fluid element's volume.

$\overleftrightarrow{r}$ and $\overleftrightarrow{\sigma}$

$$r_{xx} = r_{yy} = r_{zz} = 0\ ,\ r_{xy} = -r_{yx} = \frac{1}{2}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) = \frac{1}{2}(\vec{\nabla} \times \vec{v})_z = \frac{1}{2}\omega_z$$

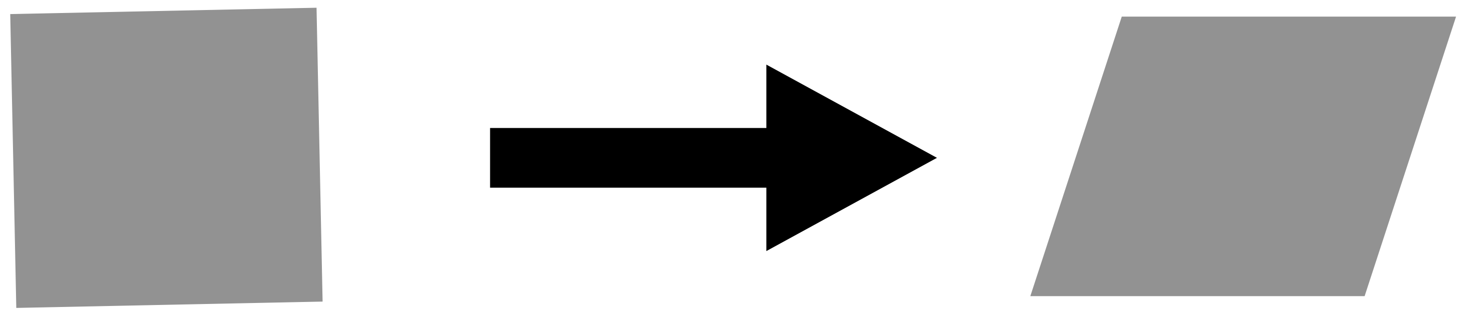
Similarly, $r_{yz} = -r_{zy} = \frac{1}{2}\omega_x\ ,\ r_{zx} = -r_{xz} = \frac{1}{2}\omega_y$



$$\overleftrightarrow{r} = \frac{1}{2}\begin{pmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{pmatrix}$$

$\overleftrightarrow{r}$ describes the local rotation of fluid.
$\overleftrightarrow{\tau}$ is symmetric but $\overleftrightarrow{r}$ is anti-symmetric. $\overleftrightarrow{\tau}$ cannot depend on $\overleftrightarrow{r}$.

$\overleftrightarrow{\sigma}$ is symmetric and trace-free. It describes the shear motion of fluid.



Bulk and Shear Viscosity

Simple model of viscosity: $\overleftrightarrow{\tau} = -\zeta\theta\overleftrightarrow{G} - 2\mu\overleftrightarrow{\sigma}$ or in component form:

$$\tau_{ij} = -\zeta\theta\delta_{ij} - 2\mu\sigma_{ij}$$

ζ : coefficient of bulk viscosity, μ : coefficient of shear viscosity.

Bulk viscosity resists the fluid's expansion and contraction.

Shear viscosity resists the fluid's shear motion.

In general, bulk viscosity $<<$ shear viscosity.

Another quantity is kinematic viscosity $\nu = \mu/\rho$

Navier-Strokes Equation

$$\boxed{\rho \frac{d\vec{v}}{dt} = \rho \left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v} \right) = - \vec{\nabla} P + \rho \vec{g} - \vec{\nabla} \cdot \overleftrightarrow{\tau}} \quad , \quad \overleftrightarrow{\tau} = -2\mu \overleftrightarrow{\sigma}$$

$$\tau_{ij} = -\mu \left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right) - \frac{2}{3} \mu \theta \delta_{ij} = -\mu \left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right)$$ for incompressible fluid ($\theta = 0$).

$$\vec{\nabla} \cdot \overleftrightarrow{\tau} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{3} \tau_{ij} \hat{x}_j \right) = -\mu \sum_{i=1}^{3} \sum_{j=1}^{3} \left(\frac{\partial^2 v_i}{\partial x_i \partial x_j} + \frac{\partial^2 v_j}{\partial x_i^2} \right) \hat{x}_j$$

$$\sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial^2 v_i}{\partial x_i \partial x_j} \hat{x}_j = \sum_{j=1}^{3} \hat{x}_j \frac{\partial}{\partial x_j} \left(\sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i} \right) = \vec{\nabla} (\vec{\nabla} \cdot \vec{v}) = 0$$ for incompressible fluid.

$$\sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial^2 v_j}{\partial x_i^2} \hat{x}_j = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} \left(\sum_{j=1}^{3} v_j \hat{x}_j \right) = \sum_{i=1}^{3} \frac{\partial^2 \vec{v}}{\partial x_i^2} = \nabla^2 \vec{v}$$

Navier-Strokes Equation for Incompressible Fluid

For incompressible fluid, $\overrightarrow{\nabla} \cdot \overleftrightarrow{\tau} = -\mu \nabla^2 \vec{v}$.

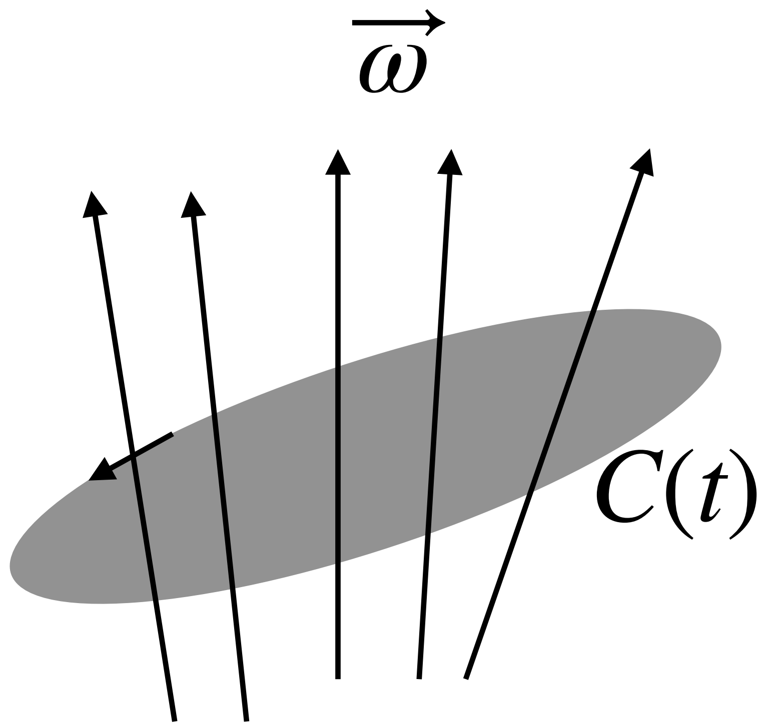
$$\rho \frac{d\vec{v}}{dt} = \rho \left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \overrightarrow{\nabla} \vec{v} \right) = -\overrightarrow{\nabla} P + \rho \vec{g} + \mu \nabla^2 \vec{v}$$

Or

$$\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \overrightarrow{\nabla} \vec{v} = -\frac{\overrightarrow{\nabla} P}{\rho} + \vec{g} + \nu \nabla^2 \vec{v}$$

$\nu = \mu / \rho$: kinematic viscosity

Evolution of Circulation



Circulation: $\Gamma(t) = \oint_{C(t)} \vec{v} \cdot d\vec{x} = \int_{S(t)} \vec{\omega} \cdot d\vec{S}$

Suppose the loop $C(t)$ follows the fluid's motion. Then

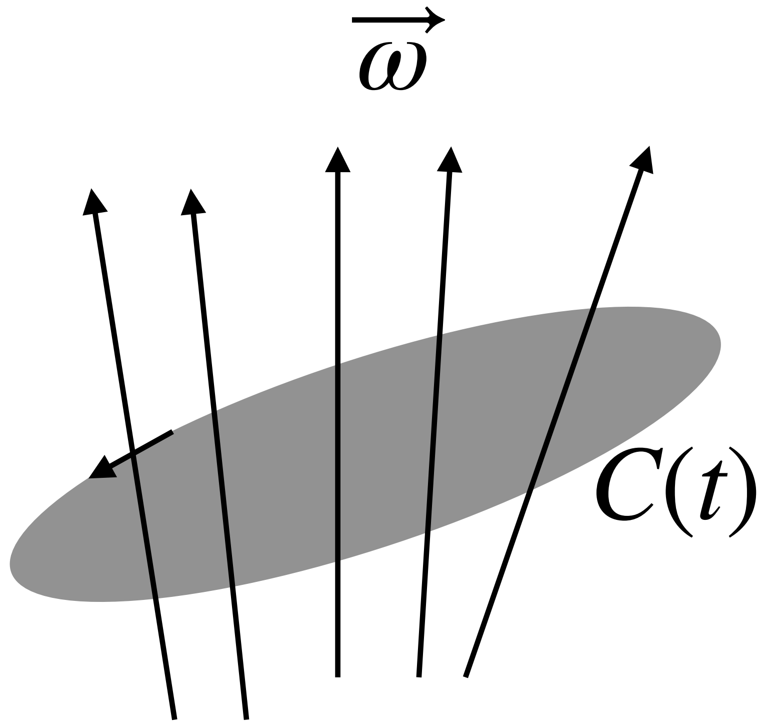
$$\frac{d\Gamma}{dt} = \oint_{C(t)} \frac{d}{dt}(\vec{v} \cdot d\vec{x}) = \oint_{C(t)} \frac{d\vec{v}}{dt} \cdot d\vec{x} + \oint_{C(t)} \vec{v} \cdot d\left(\frac{d\vec{x}}{dt}\right)$$

$$\oint_{C(t)} \vec{v} \cdot d\left(\frac{d\vec{x}}{dt}\right) = \oint_{C(t)} \vec{v} \cdot d\vec{v} = \frac{1}{2}\oint_{C(t)} dv^2 = 0$$

Navier-Stokes equation: $\dfrac{d\vec{v}}{dt} = -\dfrac{\vec{\nabla} P}{\rho} + \vec{g} - \dfrac{1}{\rho}\vec{\nabla} \cdot \overleftrightarrow{\tau}$

$$\frac{d\Gamma}{dt} = -\oint_{C(t)} \frac{\vec{\nabla} P}{\rho} \cdot d\vec{x} + \oint_{C(t)} \vec{g} \cdot d\vec{x} - \oint_{C(t)} \frac{1}{\rho}(\vec{\nabla} \cdot \overleftrightarrow{\tau}) \cdot d\vec{x}$$

Kelvin's Circulation Theorem



$$\oint_{C(t)} \vec{g} \cdot d\vec{x} = \int_{S(t)} (\vec{\nabla} \times \vec{g}) \cdot d\vec{S} = -\int_{S(t)} (\vec{\nabla} \times \vec{\nabla} U) \cdot d\vec{S} = 0$$

$$-\oint_{C(t)} \frac{\vec{\nabla} P}{\rho} \cdot d\vec{x} = -\int_{S(t)} \left(\vec{\nabla} \times \frac{\vec{\nabla} P}{\rho} \right) \cdot d\vec{S} = \int_{S(t)} \frac{\vec{\nabla}\rho \times \vec{\nabla} P}{\rho^2} \cdot d\vec{S}$$

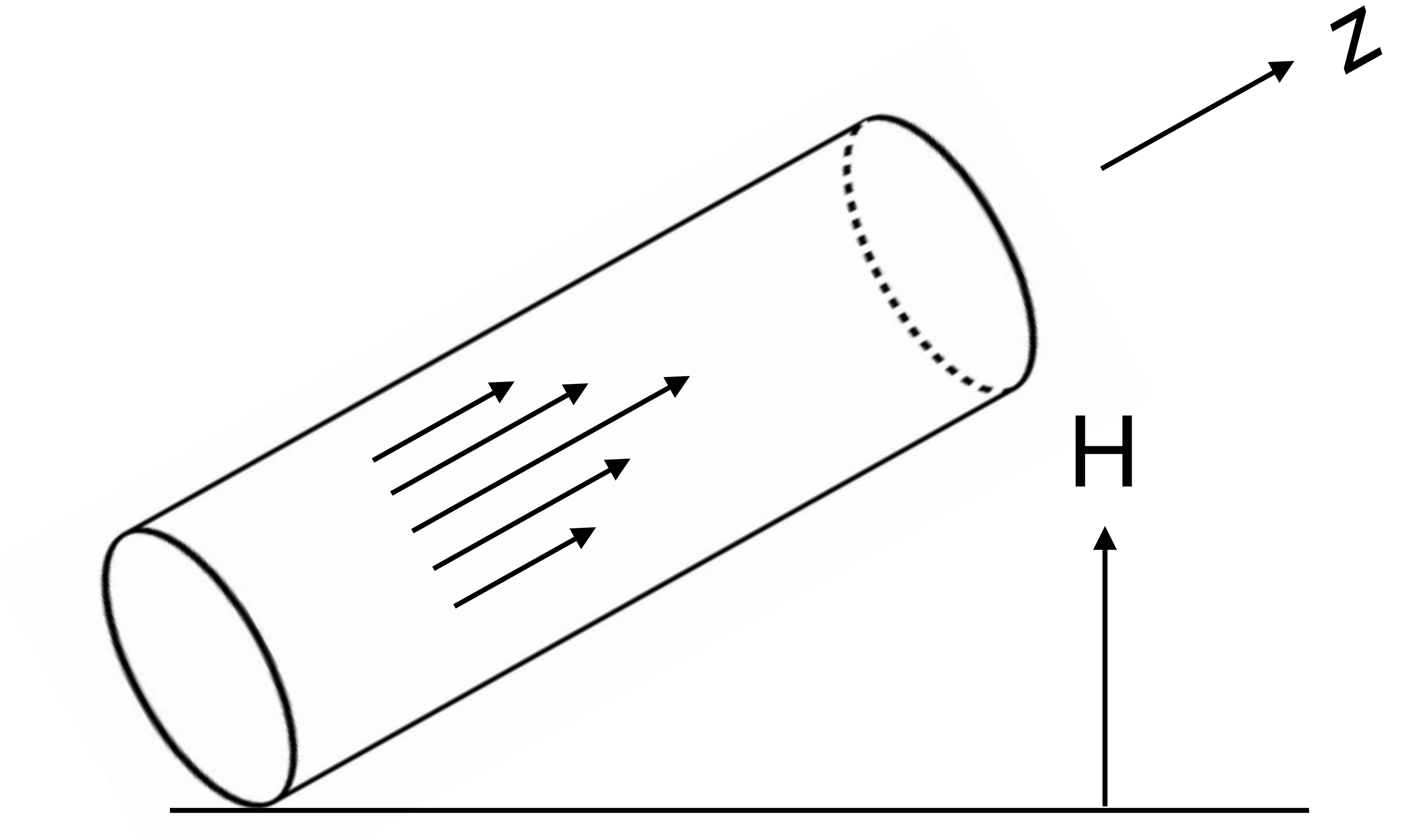
$$\frac{d\Gamma}{dt} = \int_{S(t)} \frac{\vec{\nabla}\rho \times \vec{\nabla} P}{\rho^2} \cdot d\vec{S} - \oint_{C(t)} \frac{1}{\rho} (\vec{\nabla} \cdot \overleftrightarrow{\tau}) \cdot d\vec{x}$$

If the fluid is barotropic: $P = P(\rho)$, $\vec{\nabla} P = \dfrac{dP}{d\rho} \vec{\nabla} \rho$ and so $\vec{\nabla} \rho \times \vec{\nabla} P = 0$.

$\dfrac{d\Gamma}{dt} = 0$ for barotropic, inviscid flow.

Water flowing through Cylindrical Pipe I

Continuity equation: $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0$



In cylindrical coordinates,

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_r)}{\partial r} + \frac{1}{r}\frac{\partial (\rho v_\theta)}{\partial \theta} + \frac{\partial (\rho v_z)}{\partial z} = 0$$

Looking for a steady solution ($\partial \rho / \partial t = 0$), axisymmetric and $v_r = v_\theta = 0$

$$\Rightarrow \frac{\partial v_z}{\partial z} = 0 \,, \ \Rightarrow v_z = v_z(r)$$

Navier-Stokes equation:

$$\rho \left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v} \right) = - \vec{\nabla} P + \rho \vec{g} + \mu \nabla^2 \vec{v}$$

Set $\partial \vec{v} / \partial t = 0$ and write $P = \rho g H + P_1$, where *H* is height from a reference point.

Water flowing through Cylindrical Pipe II

$$P = \rho g H + P_1 \quad \Rightarrow \quad \vec{\nabla} P = \rho g \hat{H} + \vec{\nabla} P_1 = -\rho \vec{g} + \vec{\nabla} P_1$$

Navier-Stokes equation becomes $\rho \vec{v} \cdot \vec{\nabla} \vec{v} = -\vec{\nabla} P_1 + \mu \nabla^2 \vec{v}$

Gravity is eliminated by the $\rho g H$ term. In the following, I will drop the subscript 1. So P means P_1 (pressure - $\rho g H$).

r-component:

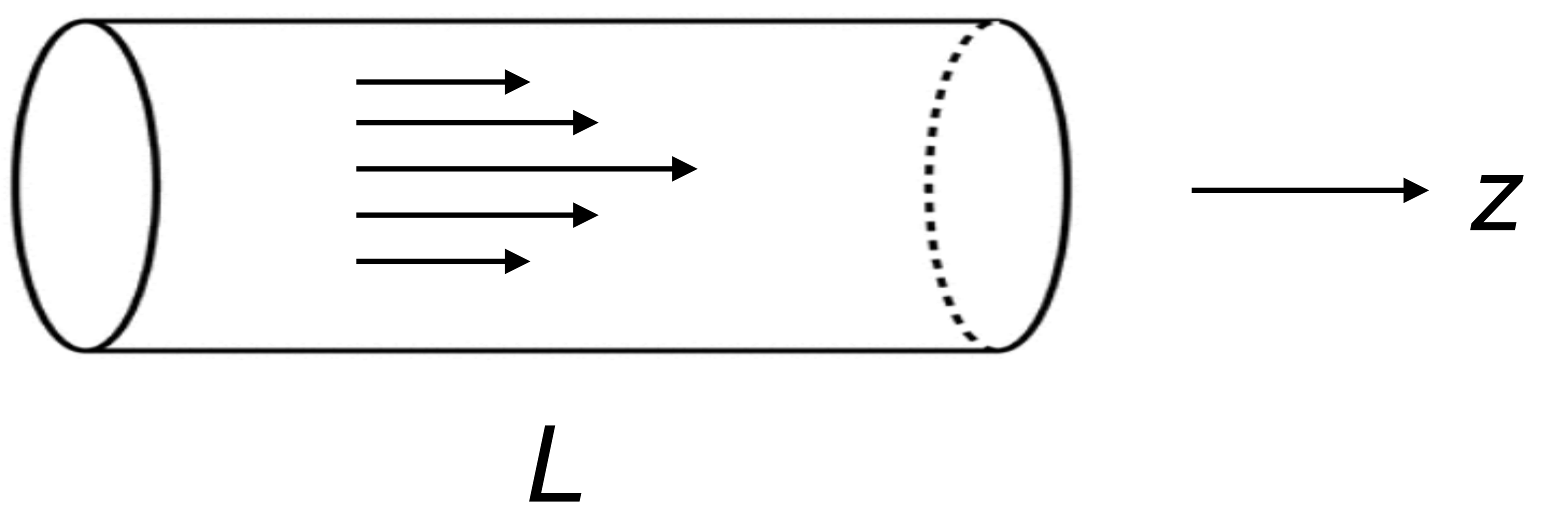
$$\rho \left(v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial P}{\partial r} + \mu \left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_r)}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2} \right]$$

$$\Rightarrow \quad \frac{\partial P}{\partial r} = 0 \; , \; P = P(z)$$

z-component: $\rho \left(v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial P}{\partial z} + \mu \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right]$

Water flowing through Cylindrical Pipe III

$$\frac{dP}{dz} = \frac{\mu}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)$$



LHS is function of *z*, RHS is function of *r*.

$$\Rightarrow \quad \frac{dP}{dz} = \frac{\mu}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = k = \text{constant}$$

Let *L* be the length of the pipe. Integrating $dP/dz = k$ from $z = 0$ to $z = L$ gives

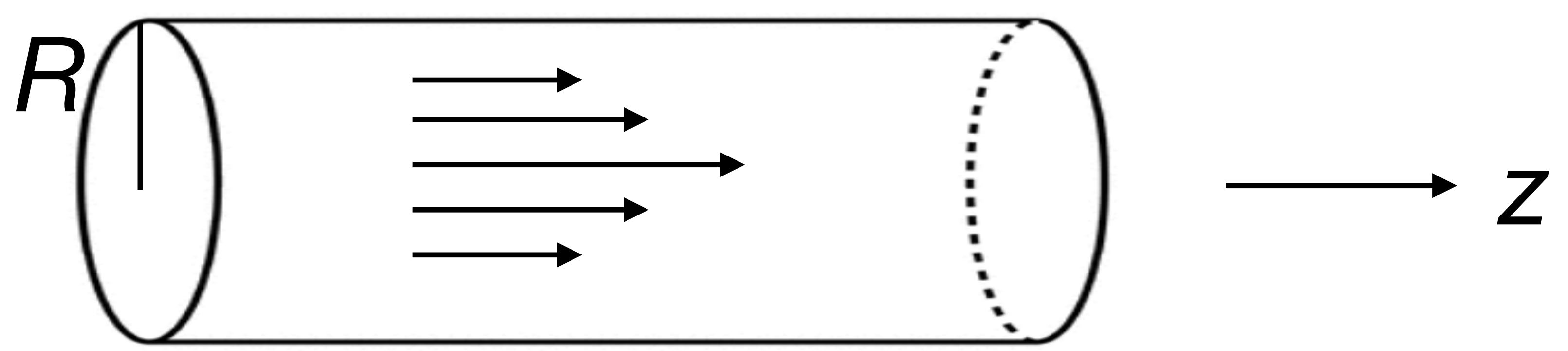
$\Delta P = kL$ or $k = -\Delta P/L$, where $\Delta P = P(0) - P(L)$ is the pressure difference between the two ends of the pipe.

$$\frac{\mu}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = -\frac{\Delta P}{L} \quad \Rightarrow \quad r\frac{dv_z}{dr} = -\frac{\Delta P}{\mu L}\int rdr = -\frac{\Delta P}{2\mu L}r^2 + C_1$$

$$v_z = \int\left(-\frac{\Delta P}{2\mu L}r + \frac{C_1}{r}\right)dr = -\frac{\Delta P}{4\mu L}r^2 + C_1 \ln r + C_2$$

Water flowing through Cylindrical Pipe IV

$$v_z(r) = -\frac{\Delta P}{4\mu L}r^2 + C_1 \ln r + C_2$$



Boundary conditions of v_z :

(1) finite at $r = 0 \quad \Rightarrow \quad C_1 = 0,$

(2) $v_z = 0$ at the wall at $r = R \quad \Rightarrow \quad C_2 = \frac{\Delta P}{4\mu L}R^2$

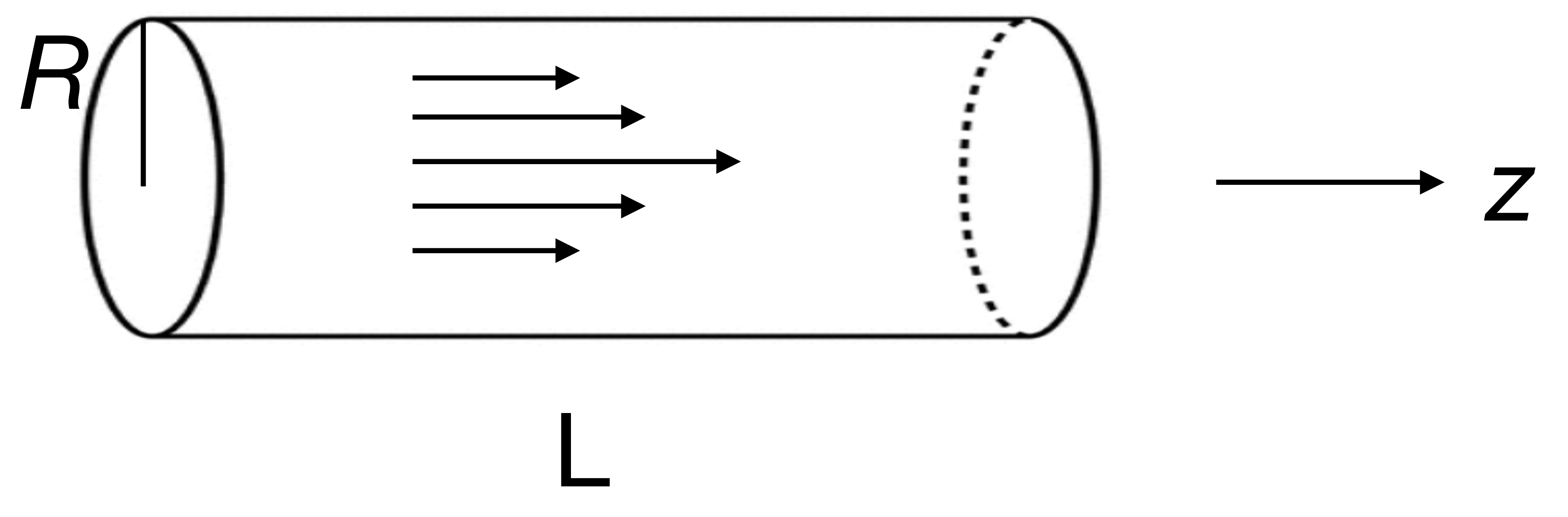
$$v_z(r) = \frac{\Delta P}{4\mu L}R^2\left(1 - \frac{r^2}{R^2}\right) \quad , \quad v_z(0) = \frac{\Delta P}{4\mu L}R^2$$

Average flow velocity is

$$\langle v_z \rangle = \frac{1}{\pi R^2}\int_0^R \int_0^{2\pi} \frac{\Delta P}{4\mu L}R^2\left(1 - \frac{r^2}{R^2}\right) r dr d\theta = \frac{\Delta P}{2\mu L}\int_0^R \left(r - \frac{r^3}{R^2}\right) dr$$

$$\langle v_z \rangle = \frac{\Delta P\, R^2}{8\mu L} = \frac{1}{2}v_z(0)$$

Water flowing through Cylindrical Pipe V



$$v_z(r) = \frac{\Delta P}{4\mu L} R^2 \left(1 - \frac{r^2}{R^2}\right)$$

$$\langle v_z \rangle = \frac{\Delta P}{8\mu L} R^2$$

Flow rate:

$$Q = \pi R^2 \langle v_z \rangle = \frac{\pi \Delta P\, R^4}{8\mu L}$$

This is called the Hagen-Poiseuille equation.

Reynolds Number and Turbulence

Navier-Stokes equation: $\rho \frac{d\vec{v}}{dt} = \rho \left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v} \right) = - \vec{\nabla} P + \rho \vec{g} + \mu \nabla^2 \vec{v}$

$$\frac{\text{inertia}}{\text{viscosity}} = \frac{\rho \, | d\vec{v}/dt \, |}{\mu \, | \nabla^2 \vec{v} \, |} \sim \frac{\rho u/T}{\mu u/L^2} \sim \frac{\rho u/(L/u)}{\mu u/L^2} = \frac{\rho u L}{\mu}$$

Reynolds number: $\mathrm{Re} = \frac{\rho u L}{\mu}$

L: characteristic length scale, *u*: characteristic speed. $T = L/u$: characteristic time.

Low Reynolds number → flow dominated by viscosity → laminar

High Reynolds number → flow dominated by inertia → turbulence

Experiments show that pipe flow only remains laminar up to Re $\sim 10^3 - 10^5$, depending on the smoothness of pipe's entrance and roughness of its walls.

Flow around Sphere with Different Re’s

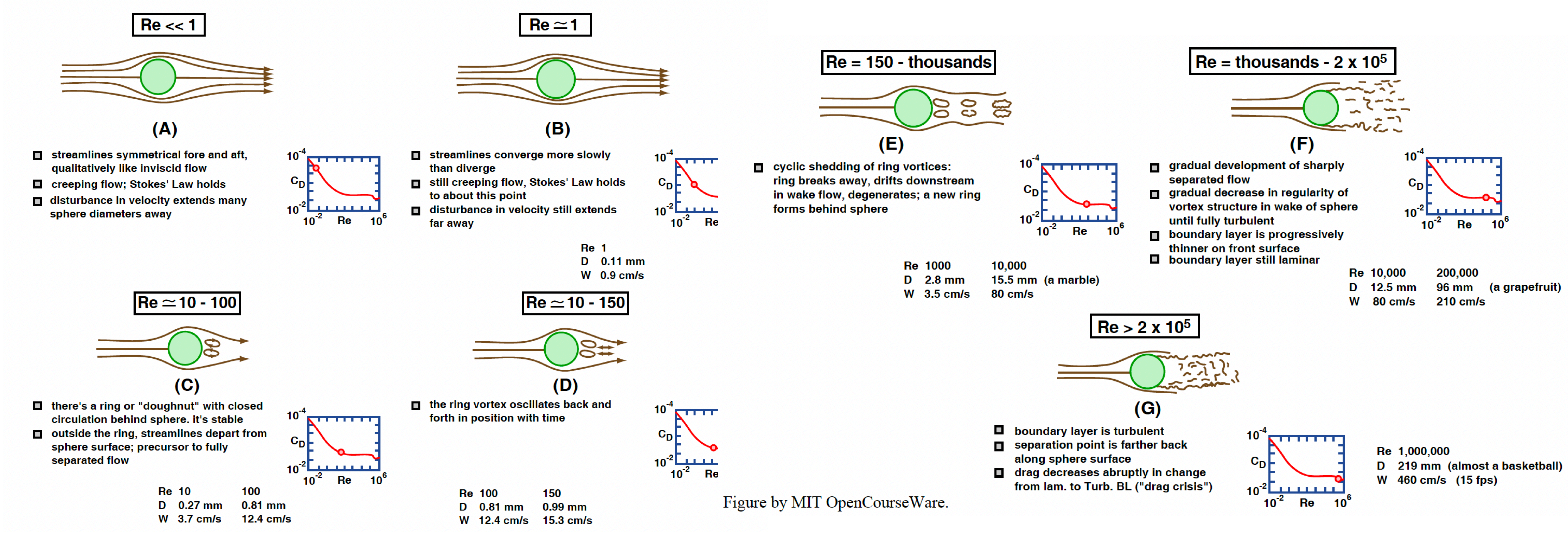


Figure by MIT OpenCourseWare.

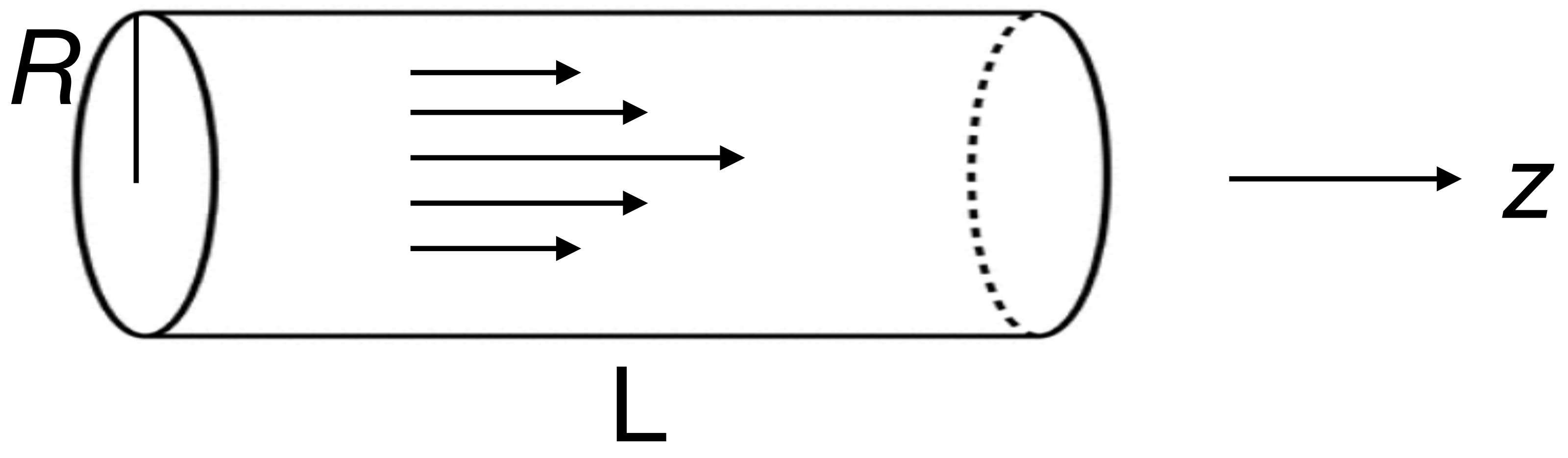
Credit: MIT OpenCourseWare

Darcy's Friction Factor and Head Loss

Hagen-Poiseuille equation: $\Delta P = \dfrac{8\mu L U_{avg}}{R^2} = \dfrac{32\mu L U_{avg}}{D^2}$

Here $D = 2R$ is the pipe diameter, $U_{avg} = \langle v_z \rangle$ is the average flow velocity in the pipe.

In the absence of viscosity, Bernoulli's equation:



$$\frac{1}{2}\rho v_1^2 + P_1 + \rho g h_1 = \frac{1}{2}\rho v_2^2 + P_2 + \rho g h_2$$

For a horizontal and steady flow, $\Delta P = P_1 - P_2 = 0$.

In the presence of viscosity, $\Delta P \propto L$. Define a dimensionless parameter called *Darcy's friction factor*:

$$\frac{\Delta P}{L} = f\frac{\frac{1}{2}\rho U_{avg}^2}{D} \quad \text{or} \quad f = \frac{\Delta P}{\frac{1}{2}\rho U_{avg}^2}\left(\frac{D}{L}\right)$$

Head loss is defined as $h_f \equiv \dfrac{\Delta P}{\rho g} \quad \Rightarrow \quad \boxed{h_f = f\dfrac{LU_{avg}^2}{2Dg}}$ (Darcy-Weisbach equation)

Darcy's Friction Factor and Head Loss (cont)

For pipes with non-circular cross section, f and h_f are defined by replacing the pipe diameter D by the *hydraulic diameter* $D_h \equiv \dfrac{4A}{P}$.

A : cross-sectional area of the pipe, P : perimeter of the pipe.

For a duct with rectangular cross section with height *h* and width *w*, $D_h = \dfrac{4wh}{2(w+h)}$.

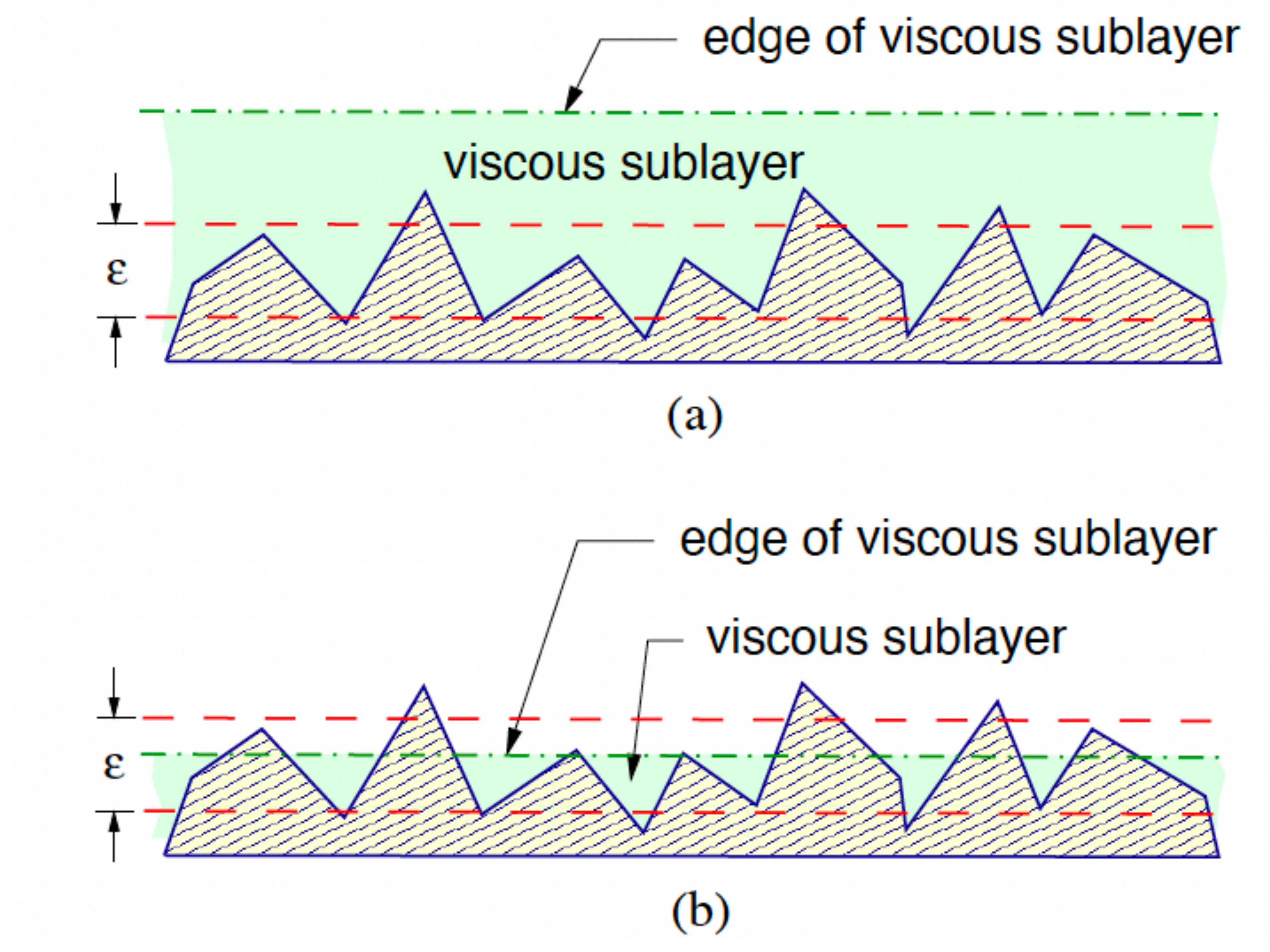
For laminar flow in a cylindrical pipe, Hagen-Poiseuille equation gives

$$f = \frac{64\mu}{\rho U_{avg} D} = \frac{64}{\mathrm{Re}}$$

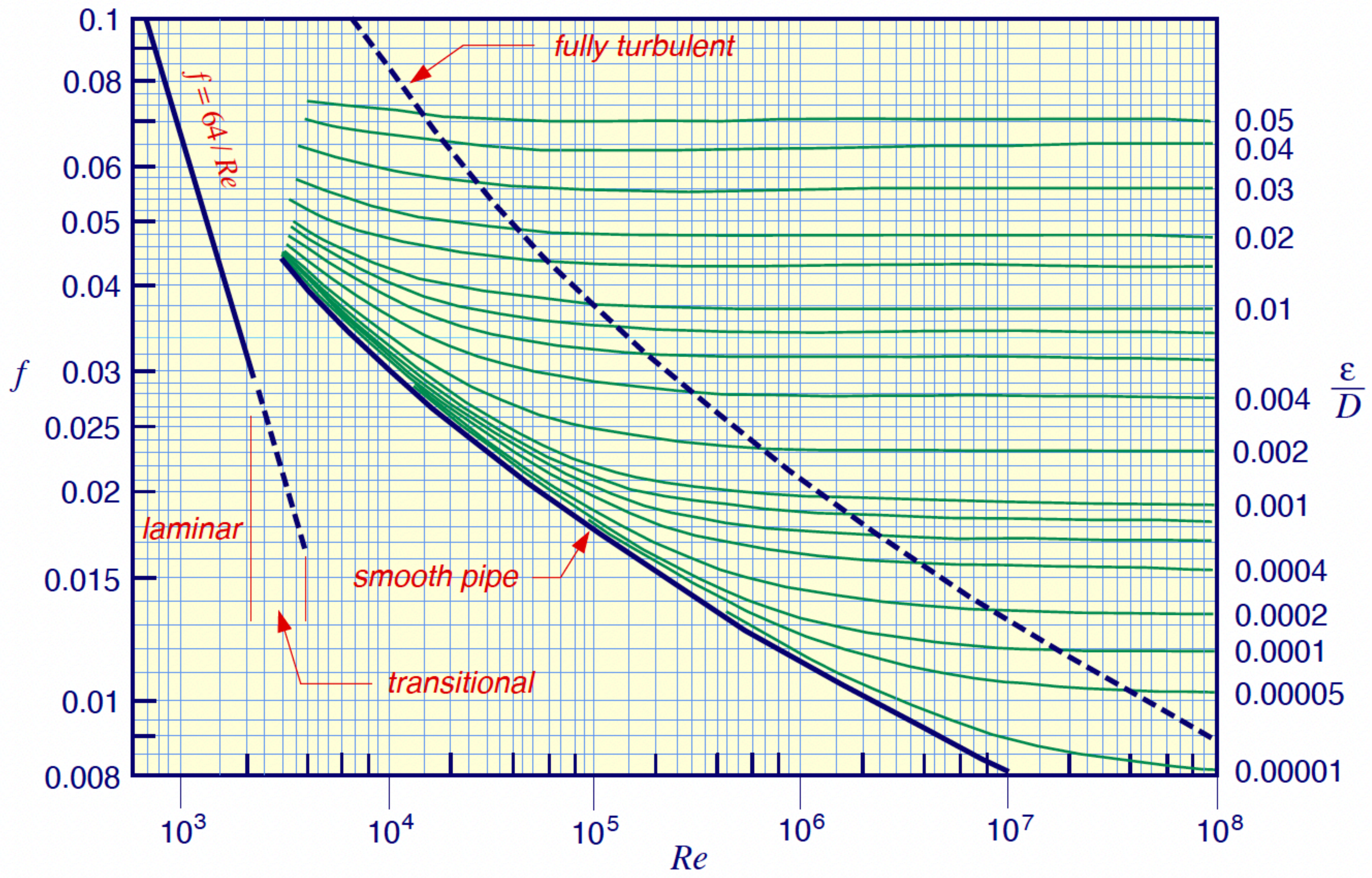
where the Reynolds number is calculated by $\mathrm{Re} = \dfrac{\rho U_{avg} D}{\mu}$.

In the presence of turbulence, f also depends on the surface roughness of the pipe ϵ.

Moody Diagram



ϵ : surface roughness of pipe



Credit: J.M. McDonough, Lectures In Elementary Fluid Dynamics: Physics, Mathematics and Applications

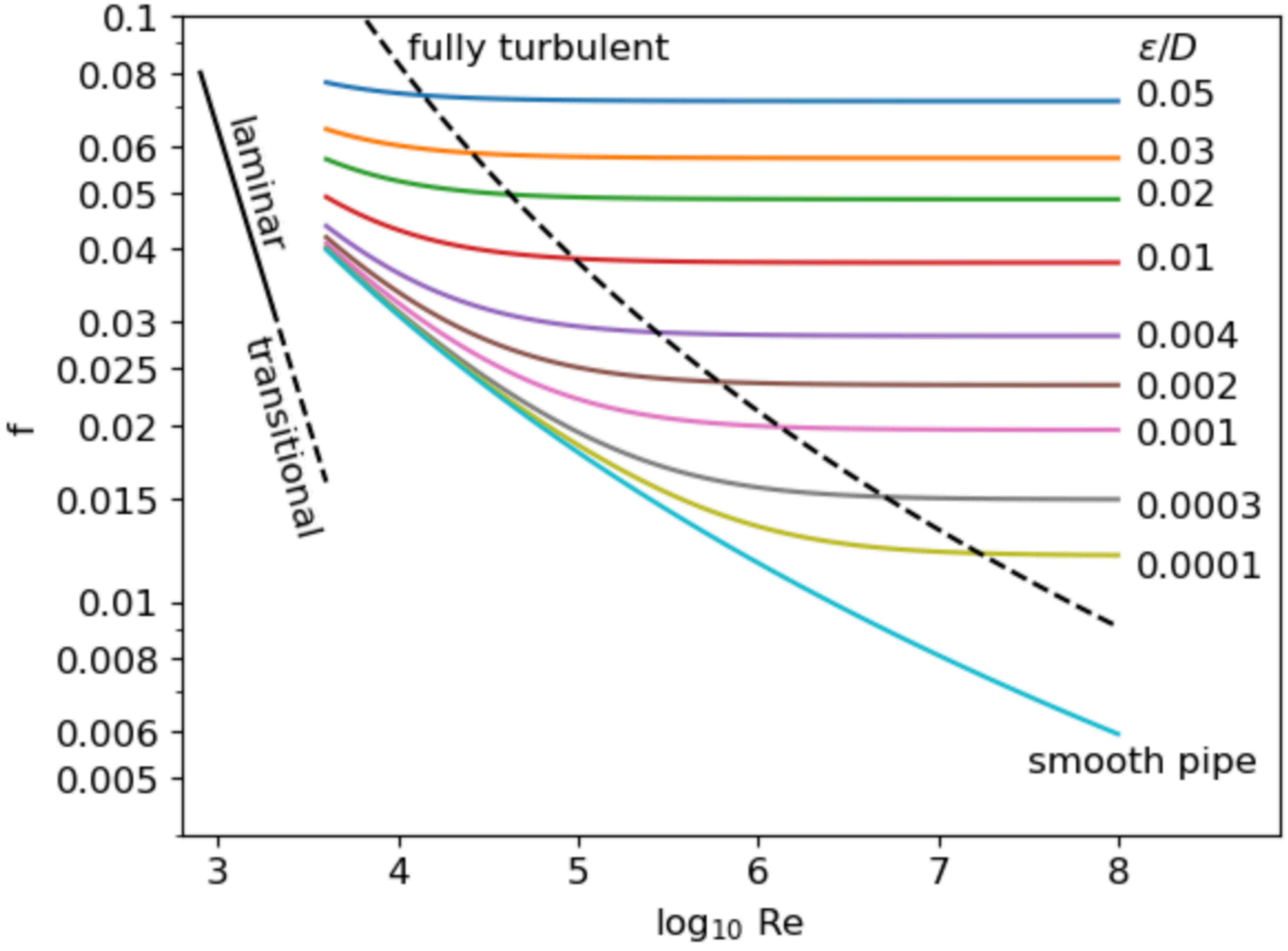
Colebrook Formula

For $4 \times 10^3 < \mathrm{Re} < 10^8$, Darcy's friction factor may be computed by the Colebrook formula

$$\frac{1}{\sqrt{f}} = -2\log_{10}\left(\frac{\epsilon/D}{3.7} + \frac{2.51}{\mathrm{Re}\sqrt{f}}\right)$$

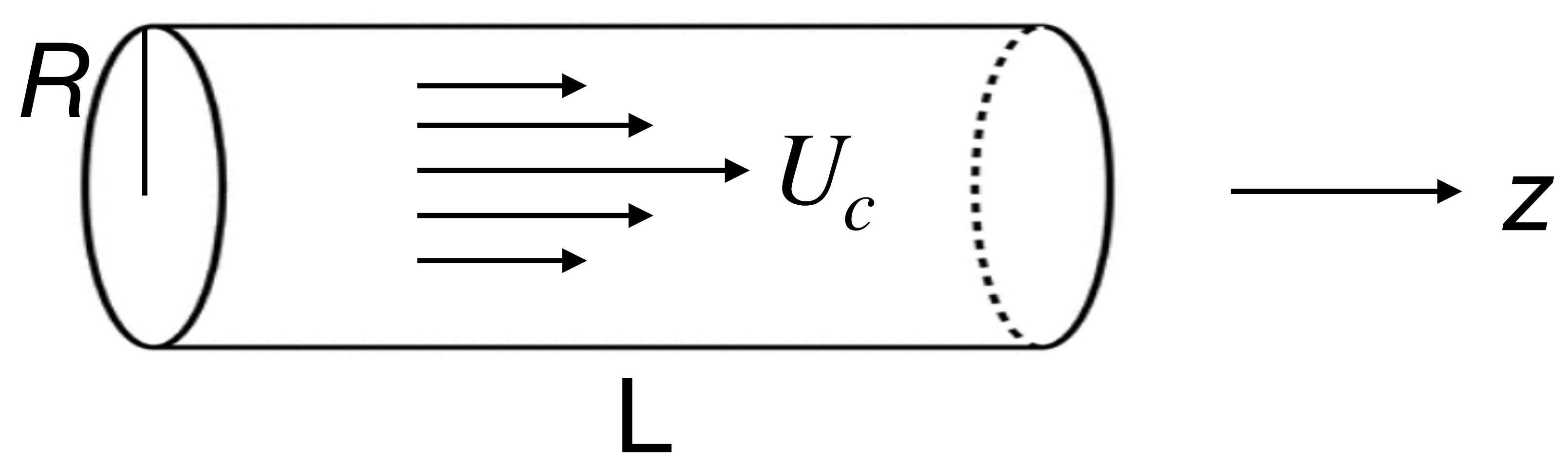
f needs to be solved iteratively.

The calculated values of f differ from experimental results < 15%.



Moody diagram calculated by the Colebrook formula

Velocity Profile

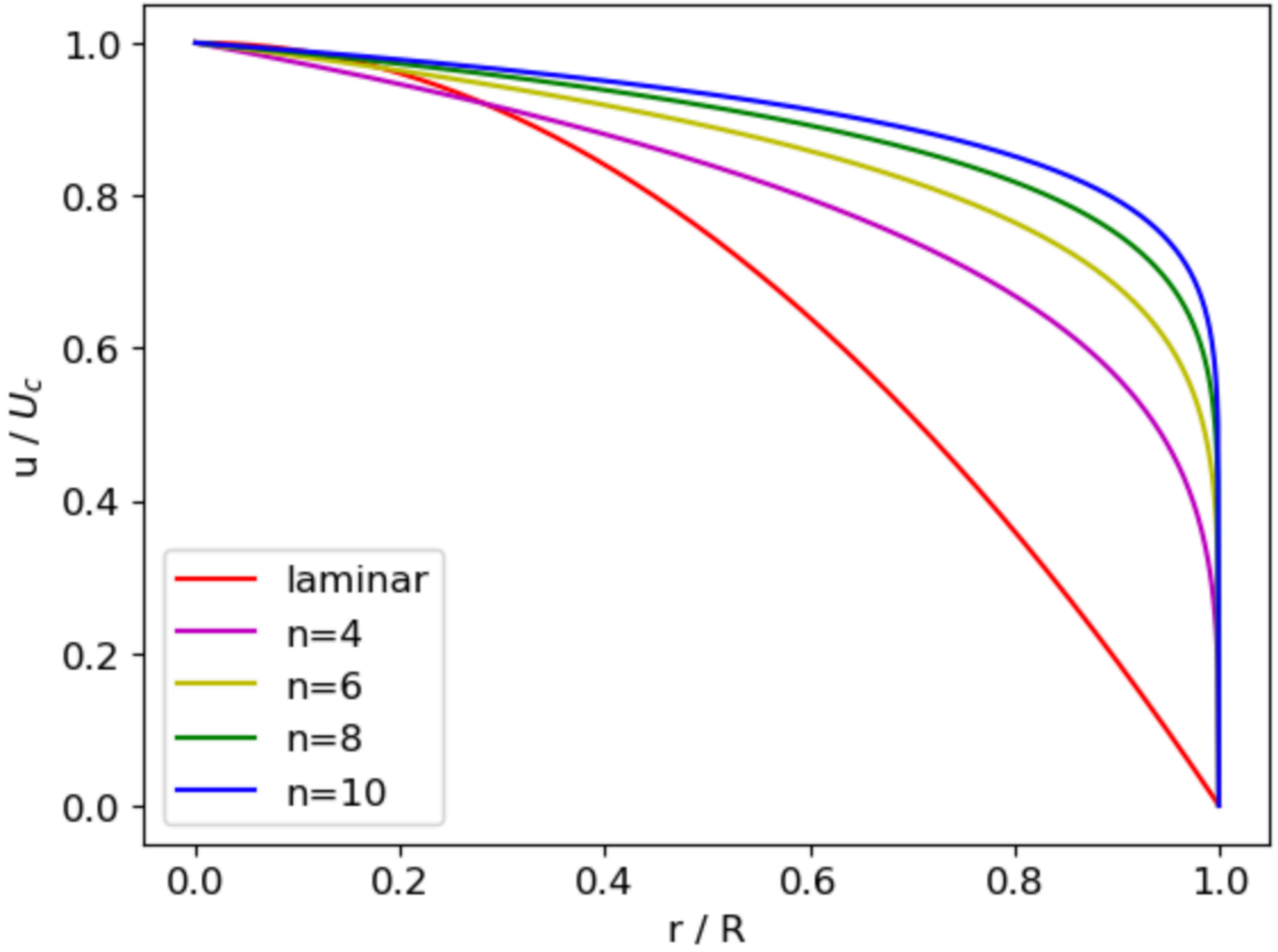


Laminar flow: $u = U_c \left(1 - \frac{r^2}{R^2}\right)$

Turbulent flow: $u = U_c \left(1 - \frac{r}{R}\right)^{1/n}$

$n = 6$ when $\mathrm{Re} \approx 2 \times 10^4$
$n = 10$ when $\mathrm{Re} \approx 3 \times 10^6$

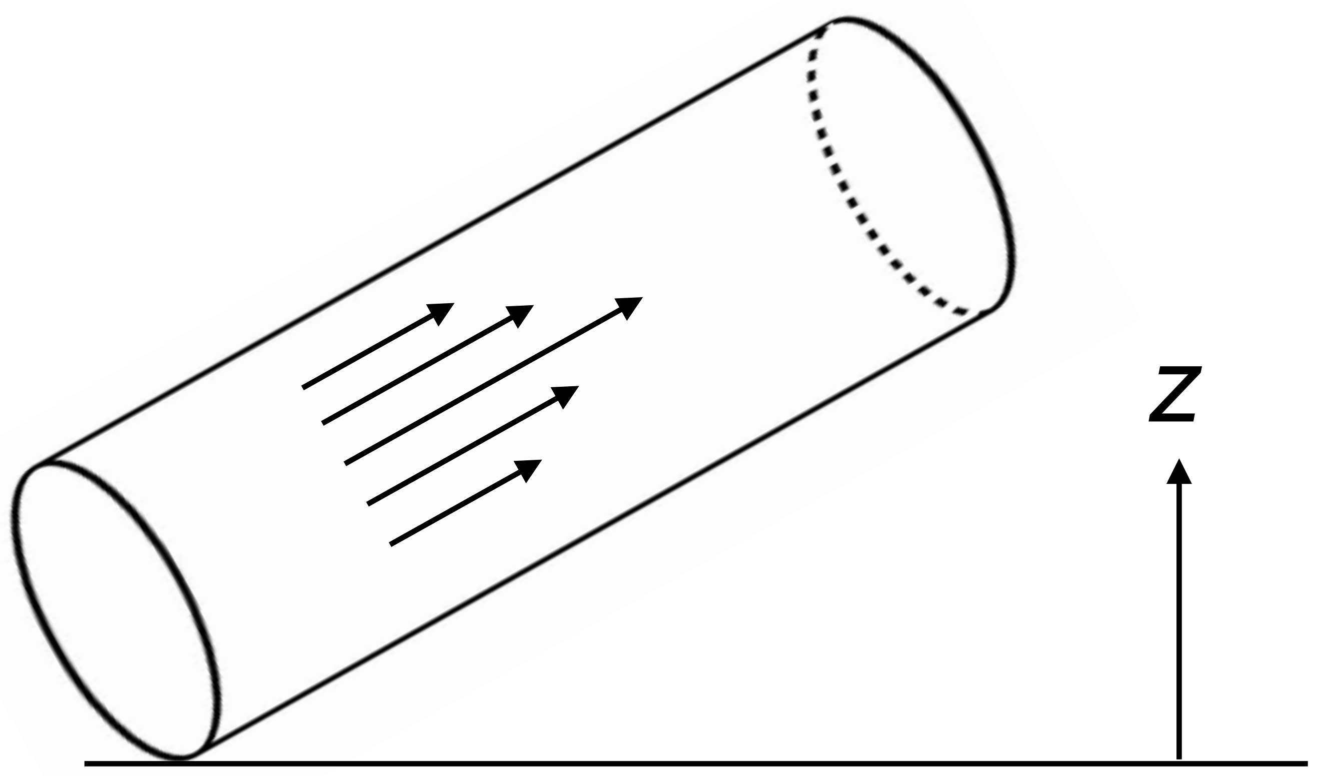
At high Re, velocity profile is relatively flat, but decreases rapidly to 0 near the wall.



Practical Head Loss Equation

Bernoulli's equation $\frac{P_1}{\rho} + \frac{1}{2}v_1^2 + gz_1 = \frac{P_2}{\rho} + \frac{1}{2}v_2^2 + gz_2$ is replaced by:

$$\frac{P_1}{\rho g} + \alpha_1 \frac{U_1^2}{2g} + z_1 + h_{pump} = \frac{P_2}{\rho g} + \alpha_2 \frac{U_2^2}{2g} + z_2 + h_f + h_{turbine}$$



U_1, U_2 : average flow speeds, α_1, α_2 : correction factor for KE.

$\alpha = 2$ for laminar flows, $\alpha \approx 1$ for turbulent flows.

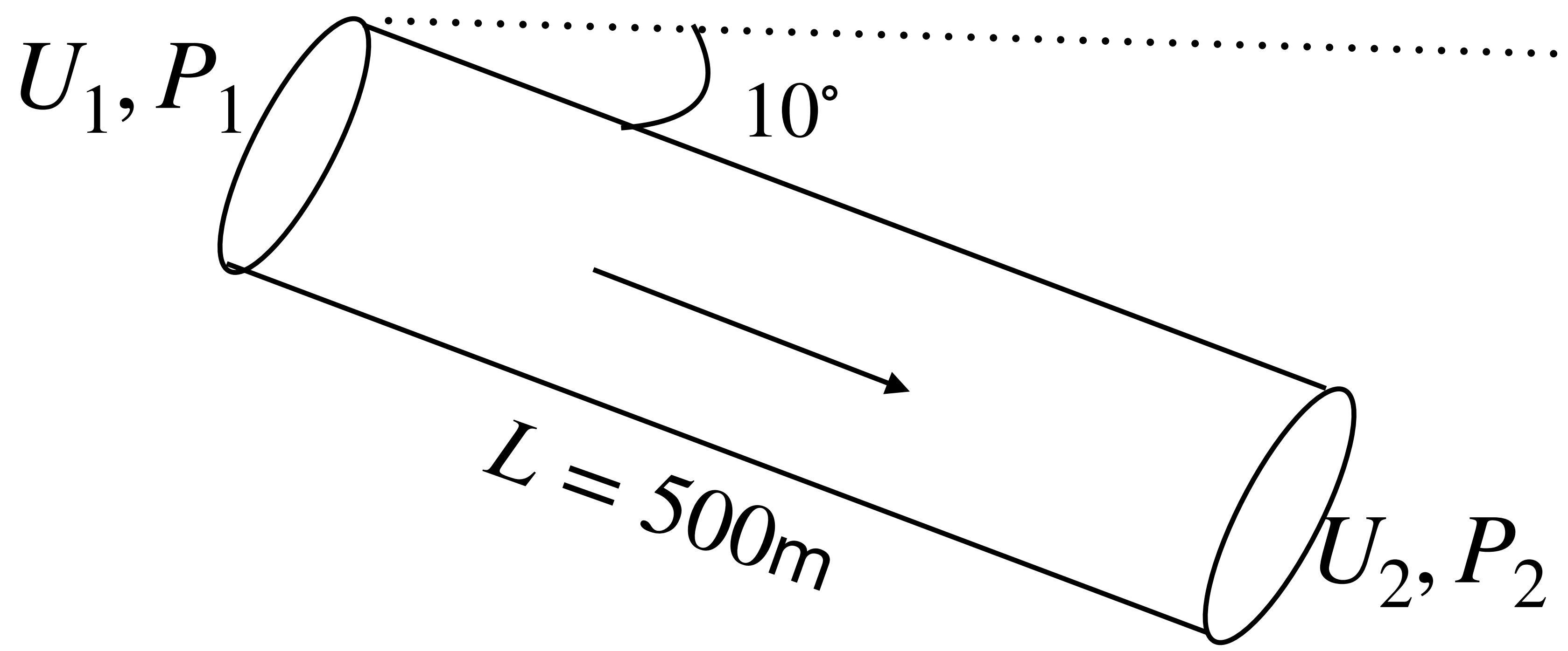
h_f : head loss caused by viscosity,

h_{pump} : head gain by a pump (if present),

$h_{turbine}$: head loss by driving a turbine (if present).

Example 1

Oil, with $\rho = 900 \text{ kg/m}^3$, and $\nu = 10^{-5} \text{ m}^2/\text{s}$, flows at $Q = 0.2 \text{ m}^3/\text{s}$ through 500 m of 0.2m-diameter cast iron pipe (roughness $\epsilon = 0.26$ mm). Determine the head loss and pressure drop if the pipe slopes down at $10°$.



Flow speeds $U_1 = U_2 = \dfrac{Q}{\pi D^2/4} = 6.37$ m/s

$$\text{Re} = \frac{\rho U D}{\mu} = \frac{UD}{\nu} = 1.27 \times 10^5$$

The flow is turbulent. Using Colebrook formula with $\epsilon/D = 0.26/200$ and the above Re, I get $f = 0.0227$. The head loss is given by the Darcy-Weisbach equation:

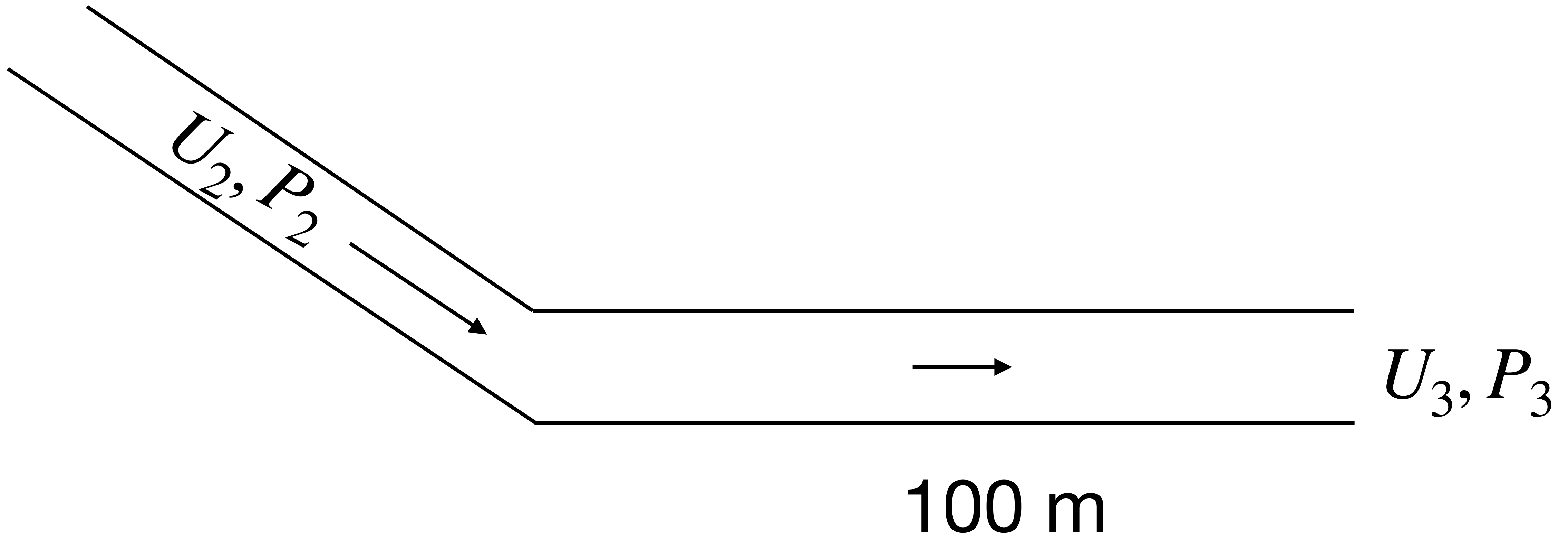
$h_f = f\dfrac{LU^2}{2Dg} = 117\text{m}.$ $\alpha \approx 1$ for turbulent flows. $\dfrac{P_1}{\rho g} + \dfrac{U_1^2}{2g} + z_1 = \dfrac{P_2}{\rho g} + \dfrac{U_2^2}{2g} + z_2 + h_f$

$$\frac{P_1 - P_2}{\rho g} = h_f - (z_1 - z_2) = 117\text{m} - (500\text{m})\sin 10° = 30\text{m}.$$

Pressure drop $\Delta P = \rho g(30\text{m}) = 2.65 \times 10^5$ Pa.

Example 2

The pipe in the previous example is connected to a horizontal pipe of length 100 m. The pipe is also made of cast iron but with diameter $D = 0.25\text{m}$. Suppose the flow rate remains the same ($Q = 0.2\text{m}^3/\text{s}$). Calculate the head loss and pressure difference in the second pipe.



$$U_3 = \frac{Q}{\pi D^2/4} = 4.07 \text{ m/s}$$

$$\text{Re} = \frac{U_3 D}{\nu} = 1.02 \times 10^5 \ , \ \epsilon/D = 0.26/250.$$

The Colebrook formula gives $f = 0.0223.$

Head loss: $h_f = f\dfrac{LU_3^2}{2Dg} = 7.54$ m.

Horizontal pipe $\Rightarrow z_2 = z_3,\quad \dfrac{P_2}{\rho g} + \dfrac{U_2^2}{2g} = \dfrac{P_3}{\rho g} + \dfrac{U_3^2}{2g} + h_f \ , \quad U_2 = 6.37$ m/s from previous calculation.

$$\Rightarrow P_2 - P_3 = \rho g h_f + \rho(U_3^2 - U_2^2)/2 = 5.6 \times 10^4 \text{ Pa}$$