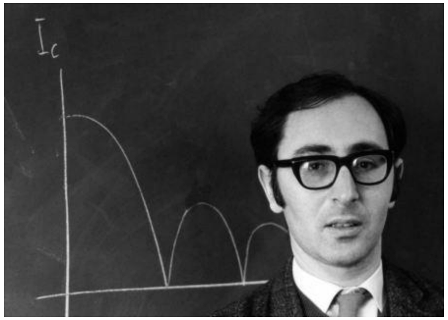
Josephson Effect

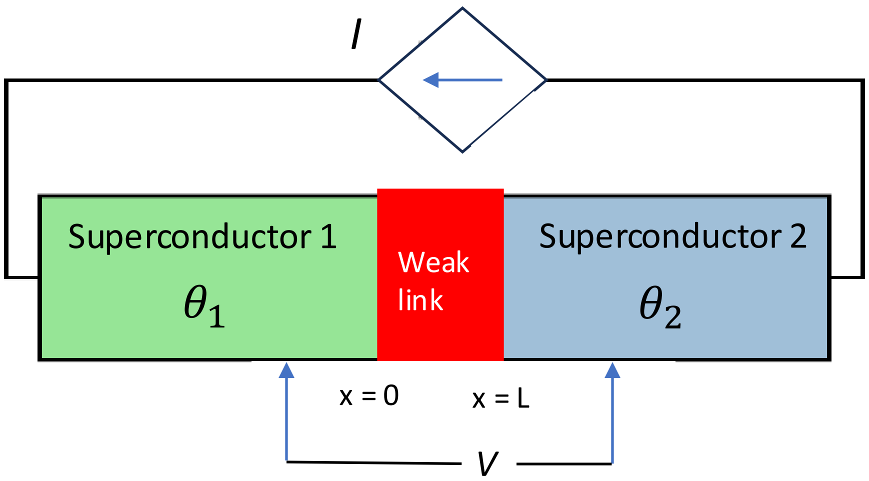
In 1962, while still a graduate student, Brian Josephson predicted what might be the most dramatic and far-reaching consequence of the long-range phase coherence in a superconductor. Consider a device in which two superconductors are separated by a *weak link*, such as a thin layer of insulating material or a normal metal – a region where bulk superconductivity doesn't exist. Each superconductor is described by a GL wavefunction of the form,

$$\psi_{1,2} = \sqrt{n_{s\,1,2}}\, e^{\,\theta_{1,2}}$$



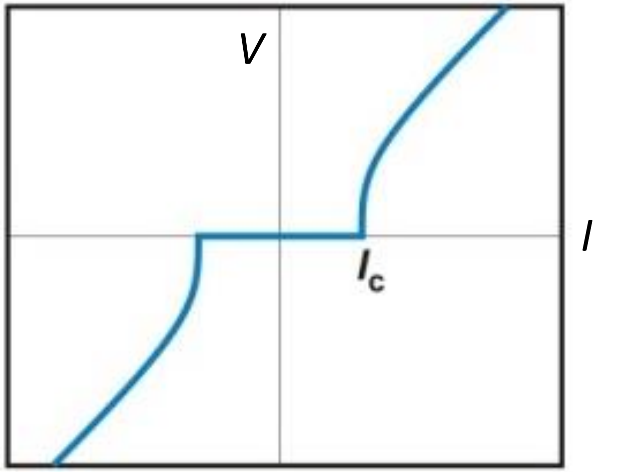
Brian Josephson

The current through the device is *I* and the voltage between the superconductors is *V*. Josephson predicted,



$$I = I_C \sin(\theta_2 - \theta_1) = I_C \sin \Delta\theta$$

$$\frac{d\,\Delta\theta}{dt} = \frac{2e}{\hbar}\, V$$

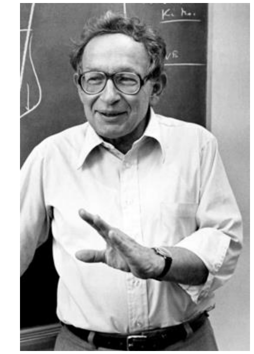


These equations imply that a supercurrent with magnitude $|I| < I_C$ will flow *even with V = 0*. I_C is called the critical current of the junction, typically in the microamp range. Ideally the current-voltage characteristic would look as shown depending on junction parameters and the circuitry used to drive them, curves may look quite different.

Discovery of the Josephson effect

John Rowell and Phil Anderson, both at Bell Telephone Labs, were the first to observe the Josephson effect. Their junction involved two different superconductors, lead and tin. A tin film was grown in an evaporator, followed by an oxide layer to form the weak link. A lead film was then grown over the oxide. The circled region of the data shows a peak current of 0.65 mA flowing at <u>zero</u> voltage. This is the Josephson effect. They also observed this current oscillate as a function of applied magnetic field. The flat region for V < 2 V represents the superconducting energy gap. For V > 2 mV the IV curve becomes ohmic, with $I \propto V$.

J.M. Rowell



P.W. Anderson

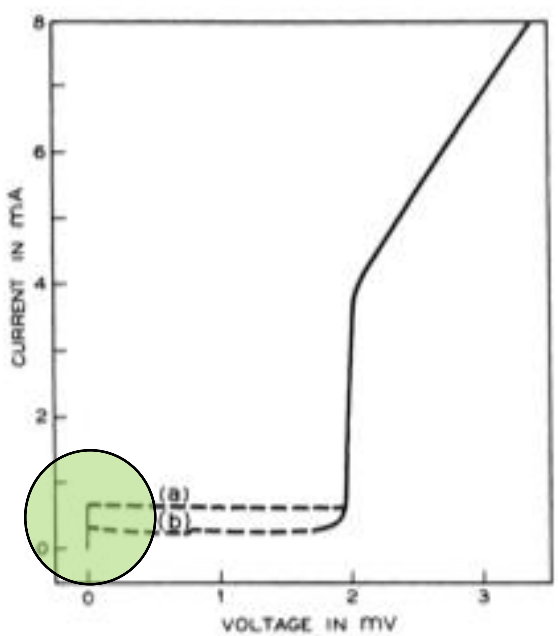
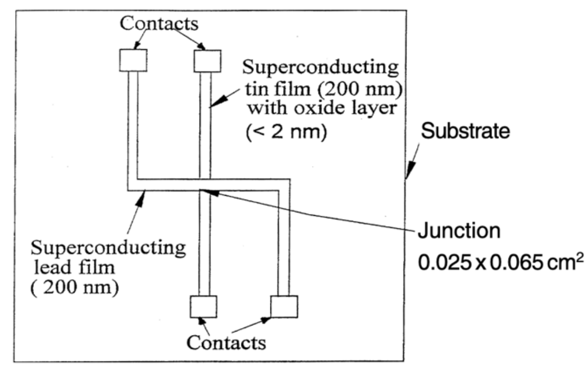


FIG. 1. Current-voltage characteristic for a tin-tin oxide-lead tunnel structure at ~1.5°K, (a) for a field of 6×10^{-3} gauss and (b) for a field 0.4 gauss.



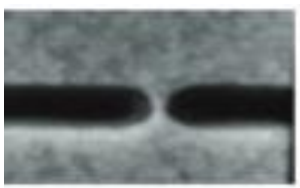
https://www.researchgate.net/publication/225810477_Great_experiments_in_physics

P.W. Anderson and J.M. Rowell, Phys. Rev. Lett. 10, 230 (1963)

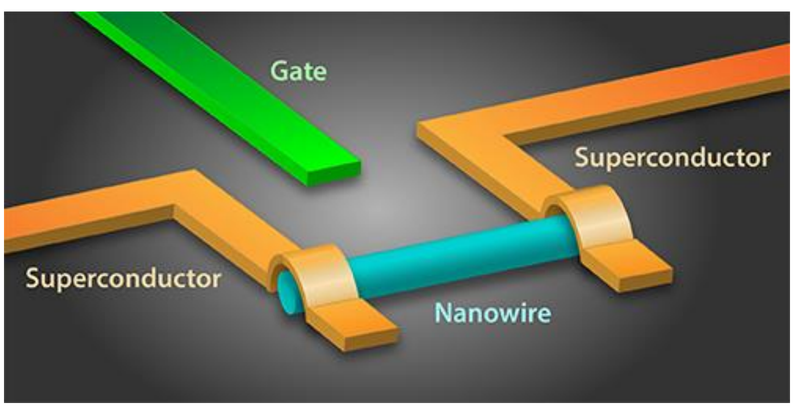
Fabrication of Josephson junctions

All kinds of weak links will show a Josephson effect. A standard method is to evaporate a superconductor like aluminum (T_C = 1.2 K), then let it oxidize to form an insulating layer, then evaporate another layer of aluminum. Al junctions are still widely used and with modern fabrication techniques, junctions can be made at the nanometer scale. Niobium (T_C = 9.2 K) is desirable for its higher transition temperature but it's harder to work with.

https://angstromengineering.com/josephson-junction-thin-film-deposition-superconducting-circuits/



Even a narrow constriction of the superconducting material (less than a coherence length) can serve as a weak link for Josephson effect. This kind of structure is called a Dayem bridge.



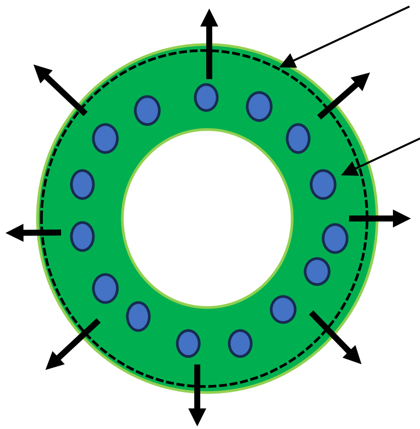
Quantum computing has focused, largely, on circuits utilizing Josephson junctions. In this case, it's desirable to control the nature of the weak link itself. The circuit shown uses a gate electrode (as in a MOSFET) to change the properties of a nanowire that serves as the weak link.

https://physics.aps.org/articles/v8/87

Josephson voltage-phase relation

One way to see the Josephson voltage-phase equation is by looking at a type II superconducting tube filled with vortices. In the figure the vortices and the trapped magnetic field are are coming up out of the page. There's an azimuthal current circulating around the perimeter of the tube that generates the B field. But, as we have learned, if there's a current flowing then vortices feel a Lorentz force that, in this case, would push them outward. The only way to keep them stationary and maintain a persistent current is to pin them with defects. However, thermal fluctuations will occasionally overcome the pinning force and vortices will slowly leak out. At any given moment the tube contains *N* vortices and therefore the total flux through the page is $\phi_B = N\phi_0$. The total phase change as we go around the perimeter of the tube is,

$$\Delta\theta = 2\pi N = 2\pi \frac{\phi_B}{\phi_0} \qquad \rightarrow \qquad \frac{d\Delta\theta}{dt} = \frac{2\pi}{\phi_0}\frac{d\phi_B}{dt}$$



Faraday's law says that $d\phi_B/dt$ is the line integral of the electric field around the loop, which is the induced EMF *V*,

$$\frac{d\Delta\theta}{dt} = \frac{2\pi}{\phi_0}\frac{d\phi_B}{dt} = \frac{2\pi}{\phi_0}\oint \vec{E} \cdot d\vec{l} = \frac{2\pi}{\phi_0} V$$

Since $\phi_0 = h/2e$ we get the Josephson relation,

$$\frac{d\,\Delta\theta}{dt} = \frac{2e}{\hbar} V$$

Current-phase relation

You can find a simple derivation of the current-phase relation in the *Feynman Lectures on Physics, Vol. III, ch. 21*. I'll follow M. Tinkham, *Introduction to Superconductivity*, Ch. 6. Start with a normalized the superconducting wavefunction,

$$g = \psi/|\psi_0|$$

g is constant in the bulk superconducting regions. In the weak link region (0 < x < L), it obeys the GL equation with two boundary conditions,

$$\xi^2 \frac{d^2 g}{dx^2} + g - g^3 = 0 \qquad g(0) = 1 \qquad , \qquad g(L) = e^{i\Delta\theta}$$

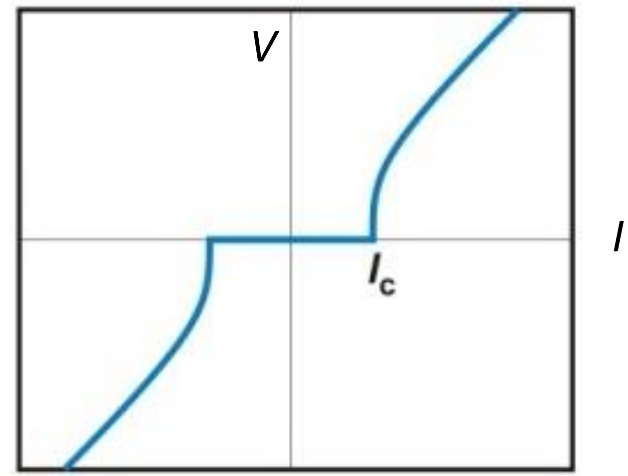
It's not immediately obvious, but if there is a finite phase difference then the first term in the equation dominates if $\xi \gg L$. Using just the first term in the differential equation, a solution satisfying the boundary conditions is,

$$g(x) \approx (1 - x/L) + (x/L)e^{i\Delta\theta\, x}$$

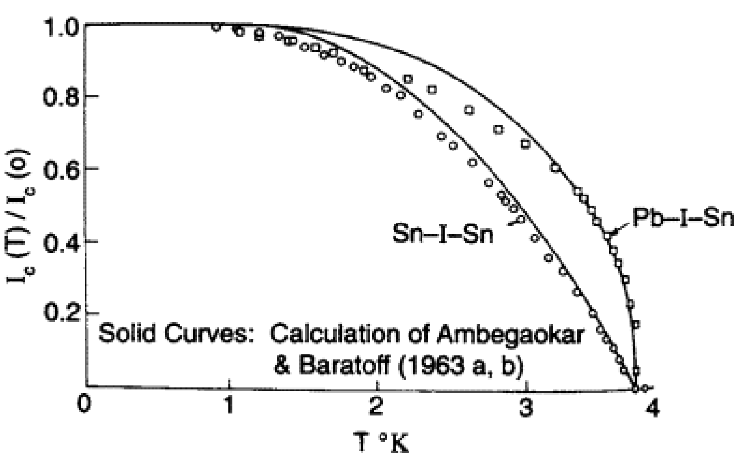
The current (in zero field) is now given by,

$$\vec{j} = \frac{\hbar q}{2mi}(\psi^*\nabla\psi - \psi\,\nabla\psi^*)$$

$$I = \left(\frac{q\hbar}{2m}\frac{Area}{L}\right)|\psi_0|^2 \sin\Delta\theta = I_C \sin\Delta\theta$$



Area is the cross-sectional area of the weak link and *q* = 2e. In its normal state, the weak link has a resistance $R_n = \rho_n L/Area$ where ρ_n is its resistivity. Junctions are often characterized by the product,



https://www.sciencedirect.com/topics/physics-and-astronomy/josephson-effect

$$I_C R_n = \left(\frac{2e\hbar}{2m}\,\rho_n\right)|\psi_0|^2$$

$|\psi_0|^2$ is proportional n_S, the number density of superconducting electrons, itself a strong function of temperature, as shown by the data on the left. A microscopic calculation by V. Ambegaokar and A. Baratoff (*Phys. Rev. Lett.* **10**, 486 (1963)) showed that $I_C R_n$ is directly related to the energy gap $\Delta(T)$ in BCS theory,

$$I_C R_n = \frac{\pi\Delta}{2e}\tanh\left(\frac{\Delta}{2k_B T}\right)$$

for Sn-I-Sn and Pb-I-Sn junctions where I stands for insulator.

AC Josephson effect

Quite early on it was recognized that the Josephson equations would lead to extremely interesting and useful effects with time-dependent voltages applied. Begin with the Josephson voltage-phase equation,

$$\frac{d\Delta\theta}{dt} = \frac{2e}{\hbar}\,V$$

Now apply a combination of AC and DC voltage: $V = V_0 + V_1 \cos\omega t$. The phase difference is,

$$\Delta\theta = \Delta\theta_0 + \frac{2eV_0}{\hbar}t + \frac{2eV_1}{\hbar\omega}\sin\omega t$$

Using the Josephson current relation and the same identity used to find the FM radio spectrum,

$$I = I_C \sin\Delta\theta = I_C \sin\left(\Delta\theta_0 + \frac{2eV_0}{\hbar}t + \frac{2eV_1}{\hbar\omega}\sin\omega t\right) = I_C \sum_{n=-\infty}^{\infty}(-1)^n J_n\left(\frac{2eV_1}{\hbar\omega}\right)\sin\left(\Delta\theta_0 + \frac{2eV_0}{\hbar}t - n\omega t\right)$$

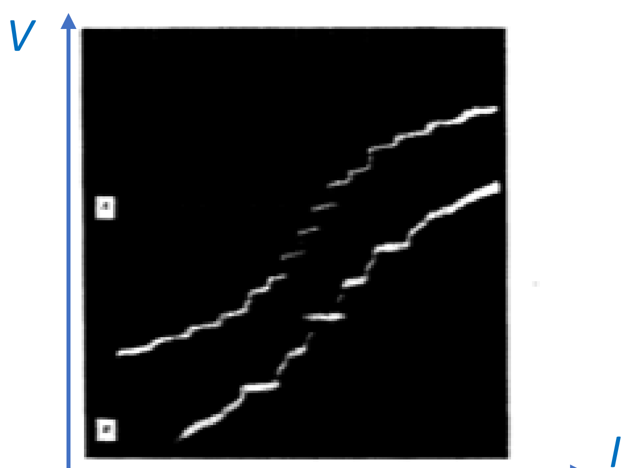


FIG. 3. Microwave power at 9300 Mc/sec (A) and 24850 Mc/sec (B) produces many zero-slope regions spaced at $h\nu/2e$ or $h\nu/e$. For A, $h\nu/e = 38.5$ μV, and for B, 103 μV. For A, vertical scale is 58.8 μV/cm, horizontal scale is 67 nA/cm; for B, vertical scale is 50 μV/cm, horizontal scale is 50 μA/cm.

At voltages given by,

$$\frac{2eV_0}{\hbar} = n\omega \quad n \in Integer$$

The current through the junction will acquire a DC value,

$$I_{DC} = I_C\, J_n\left(\frac{2eV_1}{\hbar\omega}\right)\sin(\Delta\theta_0)$$

Where the phase different $\Delta\theta_0$ depends on initial conditions. That implies that when the voltage-frequency relation is obeyed for any *n*, the DC current can vary between

$$\pm I_C\, J_n\left(\frac{2eV_1}{\hbar\omega}\right)$$

leading to *constant current plateaus* seen for each value of *n*. These are called *Shapiro steps*, after their discoverer. (S. Shapiro, *Phys. Rev. Lett.* **11**, 80 (1963))

In 1969 Parker et. al. reported extremely careful measurements on a variety of junctions verifying the universal relation,

$$\frac{2eV_0}{\hbar} = \omega \quad \rightarrow \quad \frac{2e}{h} = \frac{f}{V_0} = 483.5976 \pm 0.0012\, MHz/\mu V$$

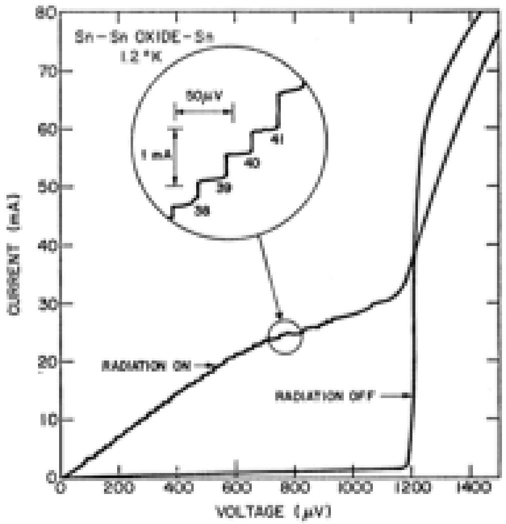


FIG. 3. *I-V* curves of a Sn-Sn oxide-Sn tunnel junction displaying radiation-induced current steps.

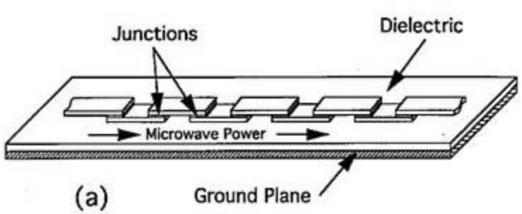
W.H. Parker, D.N. Langenberg, A. Denenstein, B.N. Taylor, Phys. Rev. ***177****, 639 (1969)*

Since the change in voltage between steps is small, the AC voltage ($V_1 \cos\omega t$) was applied by irradiating the junction with 10 GHz microwaves. Some of their data is shown on the left. By measuring the step locations using a precise voltage standard the authors could determine *e/h* with very high precision and also verify the universality of the Josephson voltage-phase relation.

By now the Josephson relations have been thoroughly verified by experiments. And since frequency measurements can be made with far more precision than voltage measurements, instead of using precise voltage measurements to test the Josephson equations, the volt is now *defined* via the Josephson effect. The voltage at the n^{th} Shapiro step is currently defined to be,

$$V_n = \frac{h}{2e} nf = \frac{nf}{483.5979\, MHz}$$

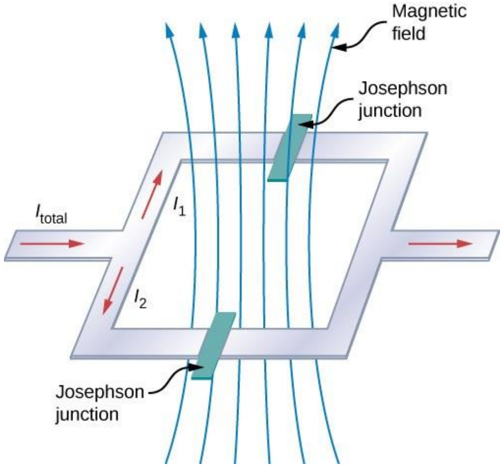
The equipment shown in the figure is a NIST programmable Josephson voltage standard. It uses an array of about 20,000 junctions irradiated with 75 GHz microwaves to generate voltages of up to 10 V with an accuracy of parts in 10^{10}.



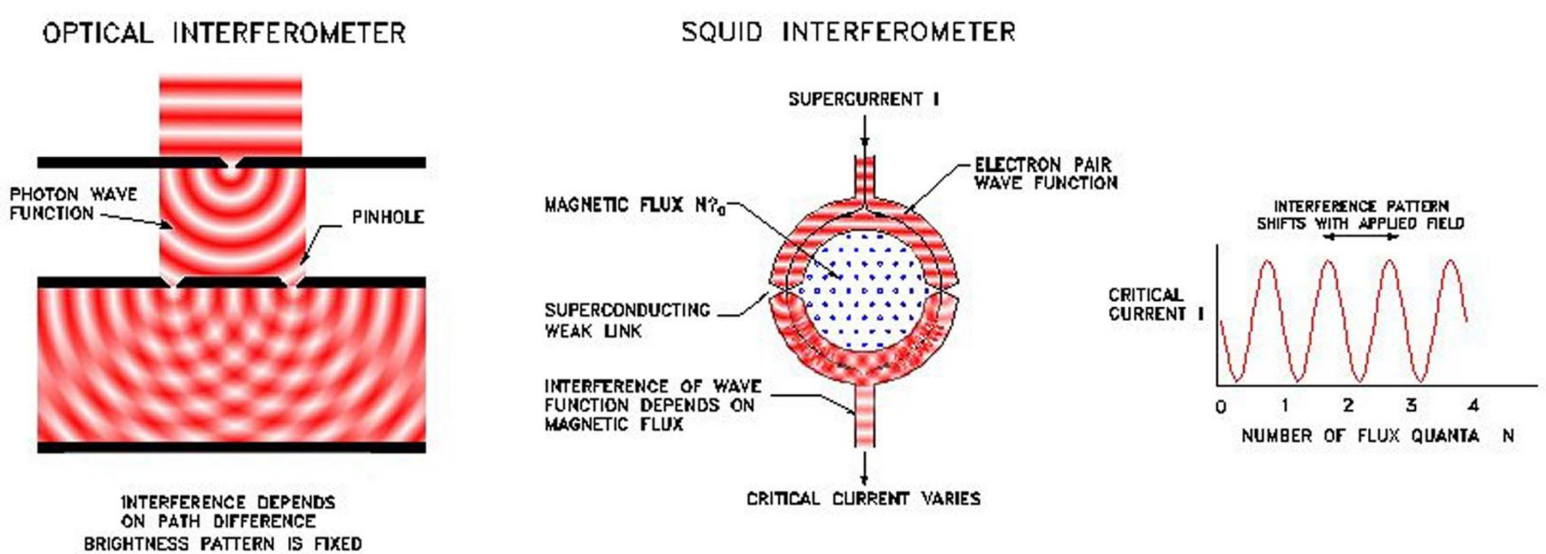
https://www.nist.gov/sri/standard-reference-instruments/sri-6000-series-programmable-josephson-voltage-standard-pjvs

The SQUID

Probably the most well-known Josephson device is shown on the right. First conceived by R. Jaklevic, J. Lambe, A. Silver and J. Mercereau in 1964. It's a loop of superconductor interrupted by 2 Josephson junctions and called a *SQUID*. That is short for *superconducting quantum interference device*. The SQUID is the ultimate magnetic field sensor, capable of detecting changes in magnetic flux of order $10^{-7}\phi_0$ and, depending on the area of the loop, magnetic fields below 10^{-15} Tesla. As the magnetic flux threading the SQUID loop increases, the critical current of the SQUID oscillates which, in turn, leads to a voltage that oscillates periodically with flux. The SQUID is analogous to a 2-slit optical interferometer, except that the the interference takes place between superconducting wavefunctions rather than light waves.



https://www.circuitbread.com/textbooks/introduction-to-electricity-magnetism-and-circuits/current-and-resistance/superconductors



https://einstein.stanford.edu/STEP/information/data/equiv2.html

Gauge invariant phase difference

Since we'll be dealing with quantum mechanics and magnetic fields we need to once more worry about gauge invariance. Physical quantities like the current must be gauge invariant. Look first at the GL expression for the current,

$$\vec{j} = \frac{q n_s}{m} \left(\hbar \nabla \theta - q \vec{A} \right)$$

Imagine we choose a new gauge defined by, $\vec{A}' = \vec{A} + \nabla \chi$. Then $\vec{j}$ will remain the same if we choose the new wavefunction phase to be $\theta' = \theta + q\chi/\hbar$. But with this gauge change the Josephson current would now become,

$$I = I_C \sin \Delta\theta' = I_C \sin\left(\theta_2 + \frac{q}{\hbar}\chi_2 - \theta_1 - \frac{q}{\hbar}\chi_1\right) = I_C \sin\left(\Delta\theta + \frac{q}{\hbar}(\chi_2 - \chi_1)\right)$$

which is not the same current we had before the gauge transformation. To fix things up we need to eliminate the term $\frac{q}{\hbar}(\chi_2 - \chi_1)$. That can be done by defining a new, *gauge-invariant phase difference*,

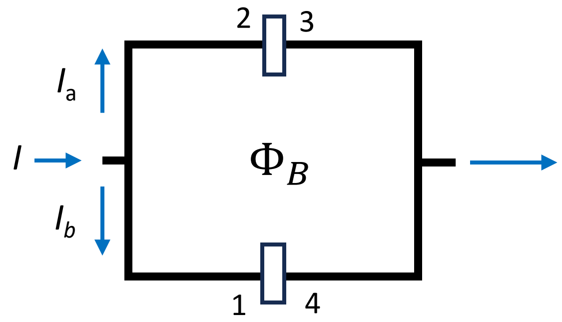
$$\Delta\gamma = \theta_2 - \theta_1 - \frac{q}{\hbar} \int_1^2 \vec{A} \cdot d\vec{l} = \theta_2 - \theta_1 + \frac{2e}{\hbar} \int_1^2 \vec{A} \cdot d\vec{l}$$

where we've used,

$$\int_1^2 \nabla \chi \cdot d\vec{l} = \chi_2 - \chi_1 \qquad q = -2e$$

The current phase relation will now be gauge–invariant if we write it as $I = I_C \sin \Delta\gamma$. What about the voltage-phase relation? The Josephson equations in a magnetic field are now,

$$I = I_C \sin \Delta\gamma \qquad\qquad \frac{d\,\Delta\gamma}{dt} = \frac{2e}{\hbar} V$$



A current $I = I_a + I_b$ flows into the SQUID and out the other side and a magnetic flux Φ_B threads the loop. Assume, for simplicity, that the critical currents are the same for the two junctions. Then the total current is,

$$I = I_C \sin \Delta\gamma_a + I_C \sin \Delta\gamma_b$$

$$\Delta\gamma_a = \theta_3 - \theta_2 + \frac{2e}{\hbar}\int_2^3 \vec{A}\cdot d\vec{l} \qquad \Delta\gamma_b = \theta_4 - \theta_1 + \frac{2e}{\hbar}\int_1^4 \vec{A}\cdot d\vec{l}$$

We also know that as we go around the entire loop the total phase change $\Delta\theta = 2\pi n$. Otherwise the wavefunction would not be single-valued. Taking a path around the loop starting at point 1 and returning, the total phase change is,

$$\oint \nabla\theta \cdot d\vec{l} = (\theta_2 - \theta_1) + (\theta_3 - \theta_2) + (\theta_4 - \theta_3) + (\theta_1 - \theta_4) = 2\pi n$$

Substituting expressions for $\Delta\gamma_a$ and $\Delta\gamma_b$,

$$(\theta_2 - \theta_1) + \left(\Delta\gamma_a - \frac{2e}{\hbar}\int_2^3 \vec{A}\cdot d\vec{l}\right) + (\theta_4 - \theta_3) + \left(-\Delta\gamma_b + \frac{2e}{\hbar}\int_1^4 \vec{A}\cdot d\vec{l}\right) = 2\pi n$$

The phase differences $(\theta_2 - \theta_1)$ and $(\theta_4 - \theta_3)$ are within the superconducting wires of the loop where the current density is,

$$\vec{j} = \frac{qn_s}{m}\left(\hbar\nabla\theta - q\vec{A}\right) = \frac{-2en_s}{m}\left(\hbar\nabla\theta + 2e\vec{A}\right)$$

If the integration path is taken many London penetration lengths from the surface of the wire, $\vec{j} = 0$ so $\hbar\nabla\theta + 2e\vec{A} = 0$. In that case,

$$\theta_2 - \theta_1 = \int_1^2 \nabla\theta \cdot d\vec{l} = -\frac{2e}{\hbar}\int_1^2 \vec{A}\cdot d\vec{l} \qquad \theta_4 - \theta_3 = -\frac{2e}{\hbar}\int_3^4 \vec{A}\cdot d\vec{l}$$

$$\rightarrow \quad -\frac{2e}{\hbar}\int_1^2 \vec{A}\cdot d\vec{l} + \left(\Delta\gamma_a - \frac{2e}{\hbar}\int_2^3 \vec{A}\cdot d\vec{l}\right) - \frac{2e}{\hbar}\int_3^4 \vec{A}\cdot d\vec{l} + \left(-\Delta\gamma_b + \frac{2e}{\hbar}\int_1^4 \vec{A}\cdot d\vec{l}\right) = 2\pi n$$

Rearranging we get,

$$\Delta\gamma_a = \Delta\gamma_b + 2\pi n + \frac{2e}{\hbar}\oint \vec{A}\cdot d\vec{l} = \Delta\gamma_b + 2\pi n + 2\pi\frac{\Phi_B}{\Phi_0} \qquad \Phi_0 = h/2e$$

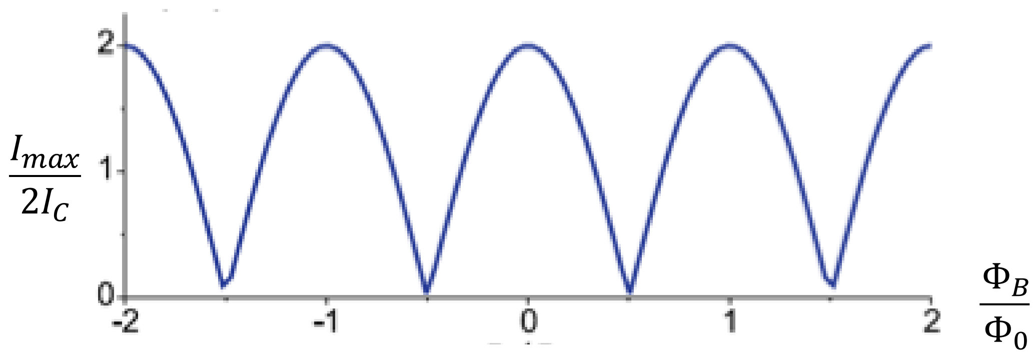
The total current is given by,

$$I = I_C \sin\Delta\gamma_a + I_C \sin\Delta\gamma_b = 2I_C \sin\left(\Delta\gamma_b + \pi\frac{\Phi_B}{\Phi_0}\right)\cos\left(\pi\frac{\Phi_B}{\Phi_0}\right)$$

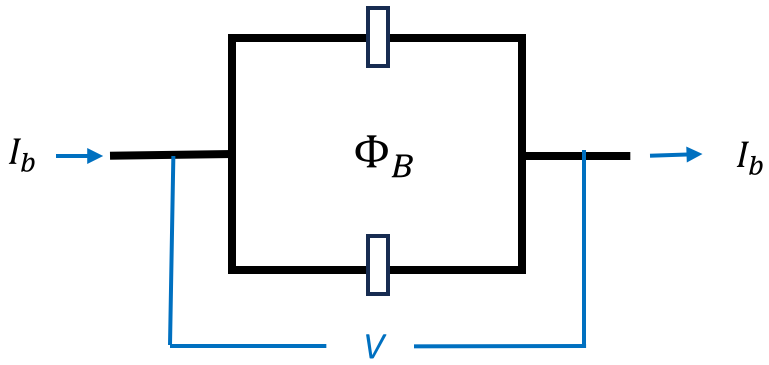
Hold the applied flux Φ_B fixed and vary $\Delta\gamma_b$ until the current is a maximum,

$$I_{max} = 2I_C\left|\cos\left(\pi\frac{\Phi_B}{\Phi_0}\right)\right|$$

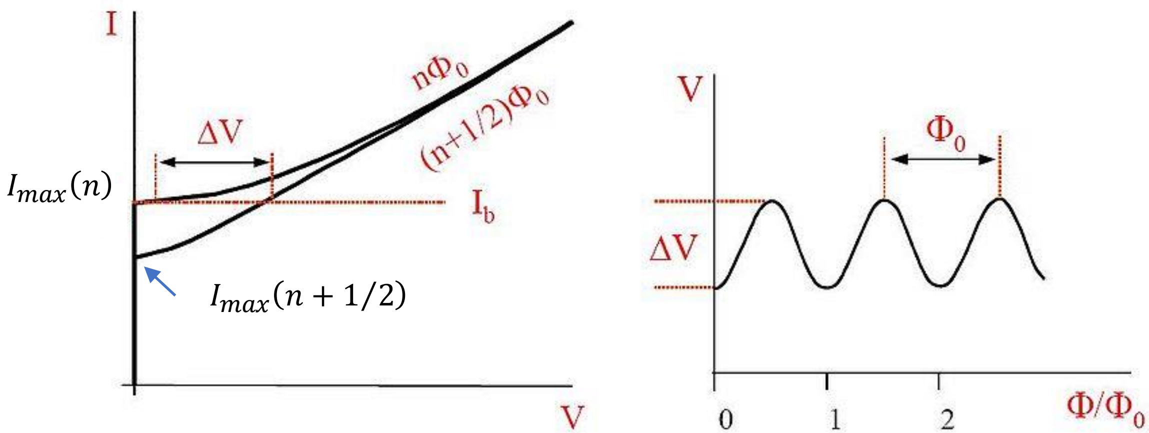
This is like a 2-slit interference pattern in which the position along the screen behind the 2 slits is replaced by the quantity $\pi\frac{\Phi_B}{\Phi_0}$. The SQUID maximum zero voltage current (i.e., its critical current) now oscillates with a period of one flux quantum.



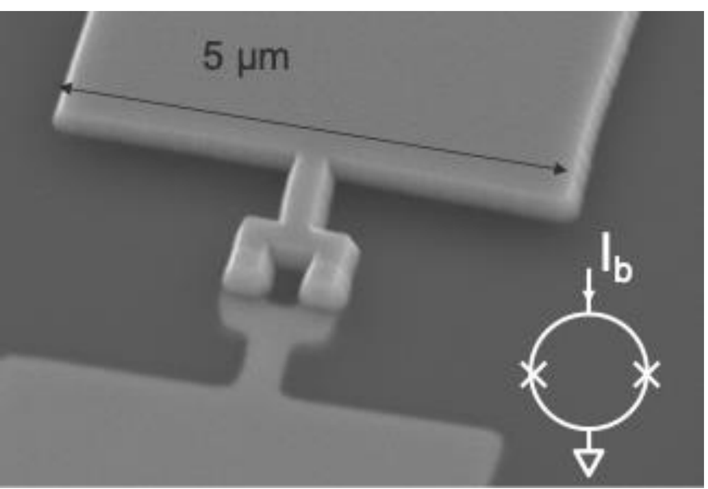
In real SQUIDs the two junctions don't have equal critical currents to I_{max} doesn't go all the way to zero but the periodicity remains. This is analogous to a 2-slit interference pattern where the two slits don't transmit equal amounts of light.



With this dependence of maximum current on the applied flux, the SQUID becomes a *flux to voltage converter*, as shown below.
Fix the total current at some value I_b. So long as $I_{max} < I_b$ the SQUID will remain in the zero-voltage state. Now increase the flux until $I_{max} > I_b$ and a voltage is developed. As you continue to increase the flux, I_{max} will periodically go above and below I_b and the SQUID voltage will oscillate with a period of one flux quantum. It's possible to measure *changes* in flux as small as $\delta\Phi \sim 10^{-7}\phi_0$. Depending on the area of the SQUID loop, fields of order 10^{-12} Tesla can be measured.

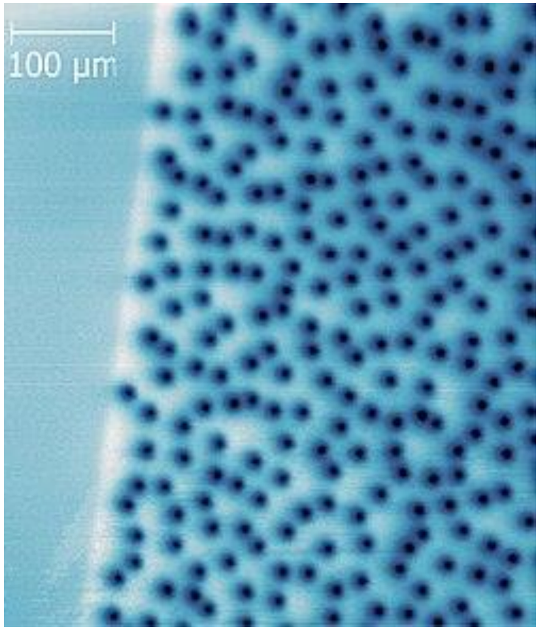


https://en.wikipedia.org/wiki/File:IV_curve.jpg



Here's a niobium SQUID made using modern nanofabrication techniques.
The tiny square in the middle is the loop

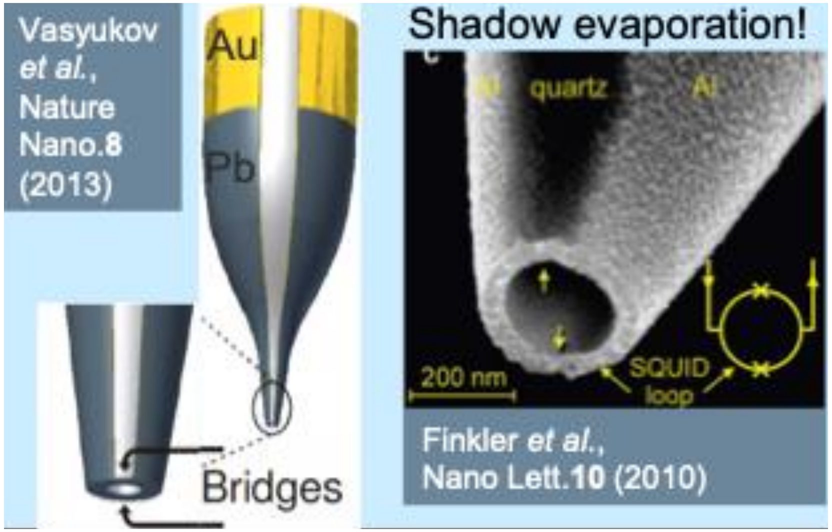
https://nanocohybri.eu/wp-content/uploads/2019/01/Koelle_2.pdf



The combination of small area SQUIDS with scanning technology makes it possible to map out the *local* magnetic response of many different materials. The figure on the left shows the arrangement of vortices in a copper oxide (high temperature) superconductor. You can see that for the right combination of pinning sites, field and temperature the arrangement of vortices is more like a glass than a periodic lattice.

https://www.nature.com/articles/srep08677

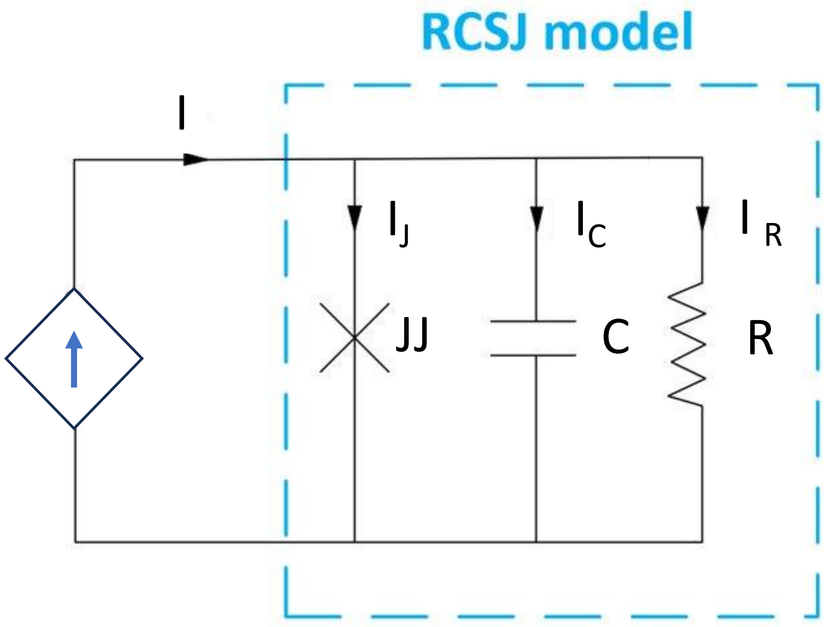
Probably the ultimate in technological sophistication is the *SQUID-on-a-tip*. Here, the SQUID loop is fabricated at the very tip of a ultrafine tip at the 200 nm scale.



https://nanocohybri.eu/wp-content/uploads/2019/01/Koelle_2.pdf

RCSJ model

A practical Josephson junction circuit must include not only the ideal Josephson characteristics but also and additional capacitance and resistance associated with the weak link. The simplest circuit model is called RCSJ (resistively-capacitively-shunted-junction), shown enclosed by the dashed lines. Assume the circuit is driven by an ideal current source I. Then Kirchoff's current law gives,



https://www.researchgate.net/publication/374765107_A_New_Behavioral-level_Model_of_Superconducting_Josephson_Junctions_with_Simulink/figures?lo=1

$$I = I_C \sin\Delta\theta + V/R + C\ dV/dt$$

Using the voltage-phase equation we can convert this into an equation for $\Delta\theta$,

$$I = I_C \sin\Delta\theta + \frac{\hbar}{2eR}\frac{d\,\Delta\theta}{dt} + \frac{\hbar C}{2e}\frac{d^2\Delta\theta}{dt^2}$$

$$I\frac{2e}{\hbar C} = \omega_p^2 \sin\Delta\theta + \frac{1}{RC}\frac{d\Delta\theta}{dt} + \frac{d^2\Delta\theta}{dt^2}$$

The left side is a constant. If it were zero, this would be the equation for a nonlinear, damped harmonic oscillator with natural frequency ω_p and quality factor Q given by,

$$\omega_p = \sqrt{2eI_C/\hbar C} \qquad Q = \omega_p RC$$

ω_p is sometimes called the Josephson plasma frequency.

We can rewrite the differential equation to look like the motion of a particle of a mass M moving along the $\Delta\theta$ axis subject to a potential energy U and a drag force,

$$M\frac{d^2\Delta\theta}{dt^2} = -\eta\frac{d\Delta\theta}{dt} - \frac{\partial U}{\partial\Delta\theta} \quad \rightarrow M \equiv \left(\frac{\hbar}{2e}\right)^2 C \qquad \eta \equiv \left(\frac{\hbar}{2e}\right)^2 \frac{1}{R}$$

$$U = -E_J \cos\Delta\theta - \frac{\hbar}{2e} I\,\Delta\theta \qquad E_J \equiv \frac{\hbar\, I_C}{2e}$$

The potential $U(\Delta\theta)$ looks like a sinusoid with with a tilted baseline – often termed a *tilted washboard* potential. Its *downward* slope increases with the applied current *I* . Begin increasing the applied current *I*.

$I = 0$

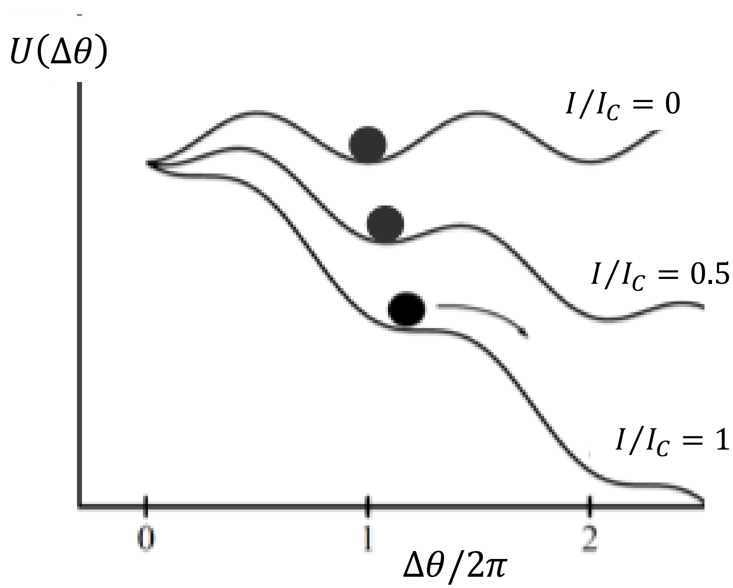
In this case there is just a static solution with $d\,\Delta\theta/dt = 0$, $V = 0$ and $I = I_C \sin\Delta\theta = 0$. The phase could also be stuck in some other potential minimum for which $\Delta\theta = 2\pi n$. Since $V = 0$ all the current flows through the junction and no current flows through *R* or *C*.

$0 < I < I_C$

The washboard begins to tilt. The "particle" moves to the right to satisfy $I = I_C \sin\Delta\theta$. The solution is still independent of time so $V = 0$ and there is still no current through R or C.

$I > I_C$

Once the current exceeds I_C *the* solutions depend on time: $d\Delta\theta/dt \neq 0$. Right at $I = I_C$ the minima in the potential have zero slope so the particle slides down the potential hill. On the flat regions it moves slowly and on the steeper portions it moves more rapidly, making a continuous series of 2π *phase slips*. The current and voltage are periodic but non-sinusoidal as a function of time.



The simplest case to analyze is a junction with small quality factor, $Q = \omega_p RC \ll 1$. It acts like a particle moving through a viscous fluid but in a sinusoidal potential. We can ignore the second time derivative so the phase difference obeys,

$$I\frac{2e}{\hbar C} = \omega_p^2 \sin\Delta\theta + \frac{1}{RC}\frac{d\Delta\theta}{dt}$$

Both the phase and therefore the junction current depends on time. The *total* current *I* is constant but the time varying *junction* current leads to a time-varying voltage whose time average can be measured with a voltmeter,

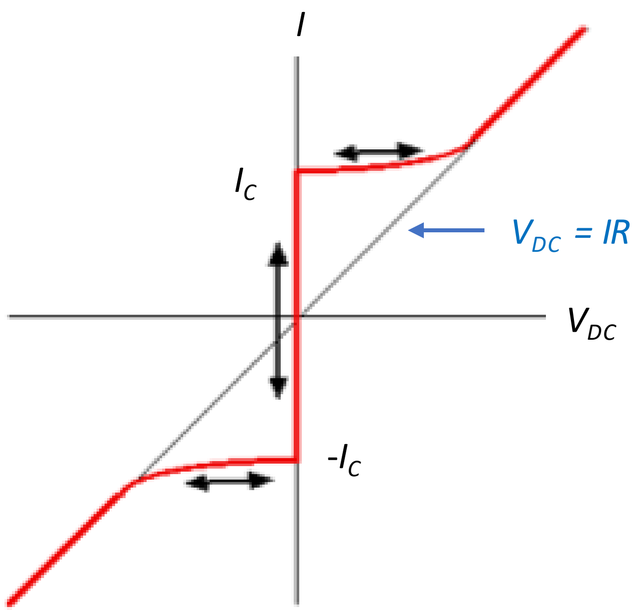
$$V_{DC} = \frac{1}{T}\int_0^T V(t)\,dt = \frac{1}{T}\int_0^T \frac{\hbar}{2e}\frac{d\,\Delta\theta}{dt}\,dt$$

To find V_{DC}, integrate the differential equation to find the time T it takes for $\Delta\theta$ to change by 2π. Then use the the Josephson voltage-phase equation to obtain the average voltage across the junction as a function of applied current,

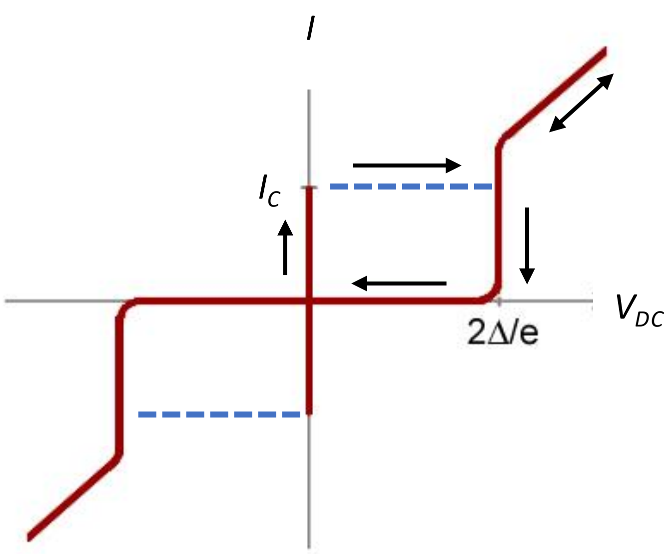
$$V_{DC} = \frac{2e}{\hbar}\frac{2\pi}{T} = R\sqrt{I^2 - I_C^2} \qquad (I > I_C)$$

The full I-V curve is shown below left. For $I < I_C$ the current is carried entirely Cooper pairs tunneling across the weak link. Once $I > I_C$, time-dependent currents and voltages develop and $V_{DC} = R\sqrt{I^2 - I_C^2}$. For $I >> I_C$ this expression extrapolates to Ohm's law.

The above derivation holds for strongly dissipative junctions ($Q << 1$) in which case the I-V curve is reversible. Junctions like this have negligible capacitance C. The weak link, rather than being an insulator, is often a short region of normal metal. When the weak link is an insulator, the junction capacitance cannot be ignored. In that case the quality factor Q can be > 1 and the I-V curve becomes hysteretic. In the limit that $Q >> 1$ the behavior is shown on the right. As the applied current increases to I_C the voltage remains at zero. Once $I > I_C$ the voltage jumps from zero to $eV = 2\Delta$, corresponding to the energy required to break a Cooper pair. As the current increases the I-V curve approaches Ohm's law. For decreasing current, once the voltage reaches $eV \approx 2\Delta$ it stays at that value until the current reaches zero and then returns to zero, as shown. For $Q > 1$ but not infinite, the returning curve intersects the vertical axis at $I_R < I_C$, known as the *retrapping current.*



https://bingweb.binghamton.edu/~suzuki/SolidStatePhysics/25-4_Josephson_junction_and_DC_SQUID.pdf



https://ourbigbook.com/cirosantilli/josephson-effect-regime

Qubits

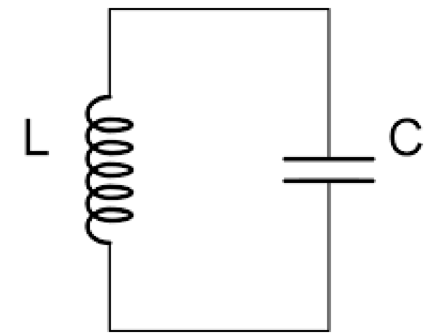
The past 30 years or so have seen an explosion of activity in quantum information science. This entails manipulating and measuring the states of many interconnected 2-level quantum systems. These 2-level systems, known as qubits, are a quantum mechanical generalization of classical logic bits of 0 or 1. If the two eigenstates of the qubit are denoted $|0\rangle$ and $|1\rangle$ the system can be in a superposition,

$$|\psi\rangle = a(t)|0\rangle + b(t)|1\rangle \qquad |a|^2 + |b|^2 = 1$$

This leads to an infinitely larger number of possible states than in the classical case. How to actually *compute* anything this way is a fascinating subject but beyond the scope of this discussion. I'll focus just on the simplest Josephson junction qubit although many other systems are being investigated as qubits for quantum computing. Just a spin ½ nucleus in a *B*-field is a perfectly fine qubit but manipulating individual spins is exceedingly difficult and impractical with present technology.

Qubit engineering requires that we treat the electrical circuits as fully quantum mechanical. Assuming it's okay to even do that, a simple example would an LC circuit whose energy is given by,

$$E = \frac{Q^2}{2C} + \frac{1}{2} L I^2$$



Treating the charge *Q* and current *I* as quantum mechanical variables, this would lead to energies,

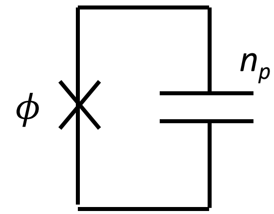
$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega_0 \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

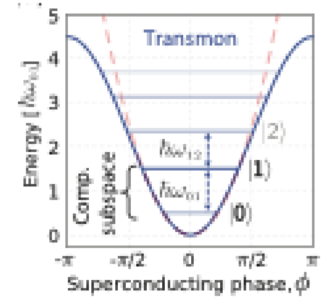
But that's *not* a two-level system, it's an infinite ladder of equally spaced energies, so a simple LC circuit can't be a qubit. What we need is something that is still dissipation-free (like an ideal inductor) but which leads to *unequal* energy spacings. That's where Josephson junctions come in. Recall from our RSCJ model that the Josephson phase difference acts like a mass moving in a tilted washboard potential,

$$U = -E_J \cos\phi - \frac{\hbar}{2e} I\phi \qquad E_J \equiv \frac{\hbar I_C}{2e}$$

This is an *anharmonic* potential. So if we replace the inductor by a Josephson junction the system will have *unequal* energy spacings.

The new circuit is shown on the right. It's called a *transmon* qubit. Since the potential energy is now anharmonic, the energy levels are no longer equally spaced. That makes it possible to treat the lowest two as a separate two-level system. In effect, the Josephson junction acts like a dissipation-free, *nonlinear* inductor. In the qubit literature it's common to denote the lowest energy state by $|0\rangle$ and the upper level by $|1\rangle$. These are called the *computational basis states*. For a spin ½ nucleus they would correspond to spin up and spin down.[1]

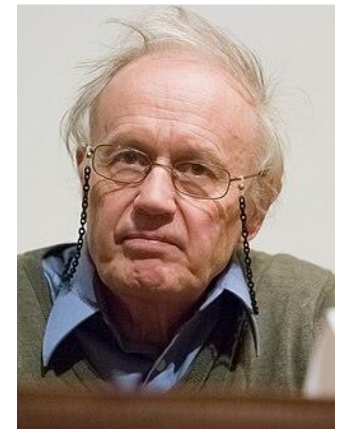




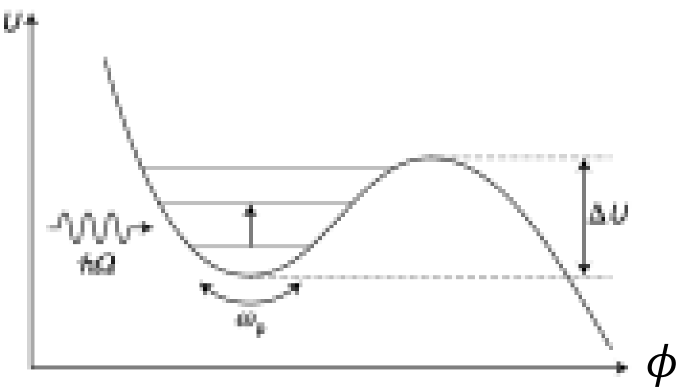
[1] This discussion is taken mostly from a review article, "A Quantum Engineer's Guide to Superconducting Qubits, P. Kranz, et. al., arXiv:1904.06560v5, (2021)

Phase as a macroscopic quantum variable

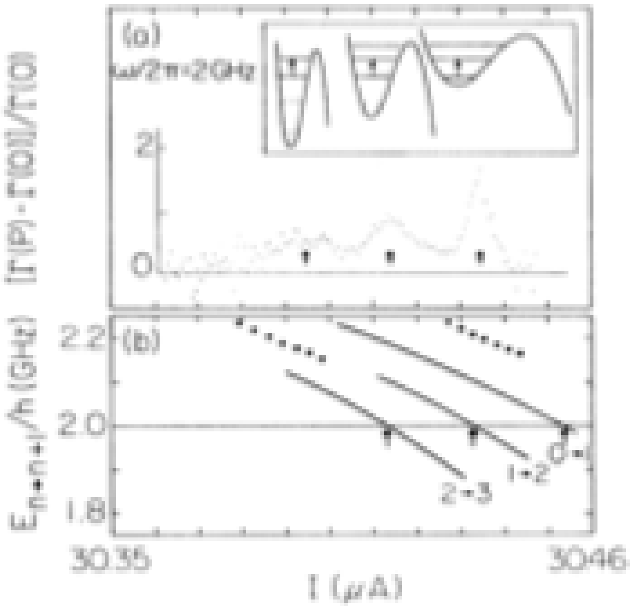
Before going on, it's not immediately obvious that the nice energy level diagram can actually be seen in the real world. Can we really treat the junction phase as a quantum mechanical variable? This idea was first promoted by Tony Leggett. He and student A. Caldeira predicted that the phase should *tunnel* through one of the barriers in the tilted washboard potential, like any good quantum system. (A.O. Caldeira and A.J. Leggett, Ann. Phys. **149**, 374 (1983)) Of course the Josephson equations *come* from the quantum mechanical behavior of a huge number of electrons acting together in a superconducting state. But unless the system is very cold, the phase acts usually like a classical variable as in the RCSJ model, albeit with thermal fluctuations. If we treat the phase as a truly quantum mechanical then there will be a series of unequally spaced energy levels in each well of the tilted washboard potential, shown below.



A.J. Leggett

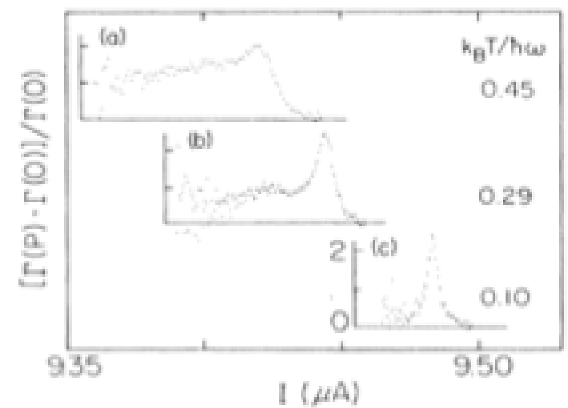


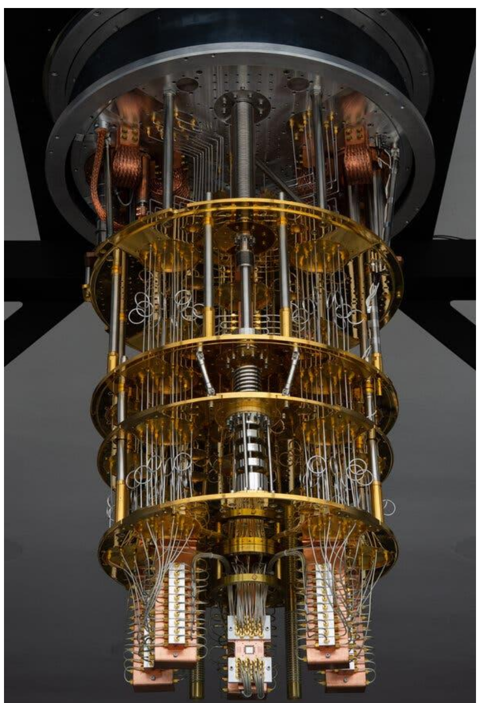
If you now irradiate the system with microwaves, this will lead to an increase of events in which the phase either jumps over or tunnels through the potential barrier and out of the zero-voltage state. This was first observed by J. Martinis, M. Devoret, J. Clark in 1985.



The Martinis et. al. experiment showed that when each of the first three energy transitions in the potential well was excited by microwaves, there was a peak in the escape rate $\Gamma(P)$ of the phase, as predicted. The microwave frequency was fixed but the energy spacing could be varied by changing the DC current bias *I*, somewhat like change the **B** field in NMR. The data below shows that as the temperature goes down, the energy levels are more and more well-defined and the phase acts more like a true quantum variable. For qubits we want only the lowest pair of energy levels but it's reassuring that they are all there.

J. Martinis, M. Devoret, J. Clark, Phys. Rev. Lett. 55, 1543 (1985)





There are many different processes that effectively broaden the Josephson junction energy levels and most of them are reduced considerably as the system gets colder. The upshot is that for our circuit to act like a true qubit we need it to be cold,

$$\frac{k_B T}{\hbar\omega_q} \ll 1$$

For typical Josephson junction qubits the frequency $f_q = \omega_q/2\pi$ is generally in the low microwave range. For a 5 GHz energy spacing and T = 20 mK, $k_B T/\hbar\omega_q \approx 0.1$. To operate electronics as such low temperature requires a *dilution refrigerator*, one example of which is shown. The lower end of this device will operate at 10 - 20 mK while the upper end will be closer to 4 K. This entire assembly will be surrounded by several layers of insulation and sits in a vacuum to reduce the heat leak from room temperature.

Assuming the junction is cold enough to treat like a true 2-level quantum system, its Hamiltonian now has the form,

$$H = \frac{Q^2}{2C} - E_J \cos\phi \approx \frac{(2e)^2}{2C}\, n_p^2 + \frac{1}{2}\, E_J \phi^2 - \frac{1}{24} E_J \phi^4 = 4E_C\, n_p^2 + \frac{1}{2}\, E_J \phi^2 - \frac{1}{24} E_J \phi^4$$

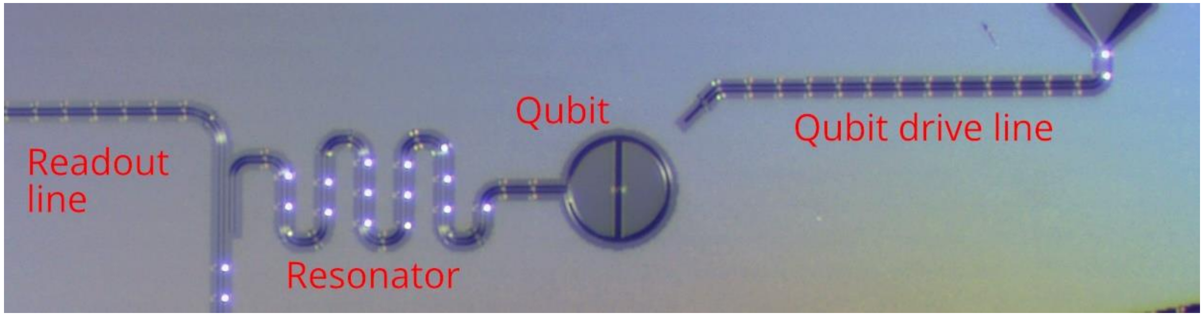
The charge variable is *2e* times the number of Cooper pairs n_p on the capacitor. The energy spacing of the two levels is given by,

$$\hbar\omega_q = \sqrt{E_C E_J} - E_C \qquad E_C = \frac{e^2}{2C}$$

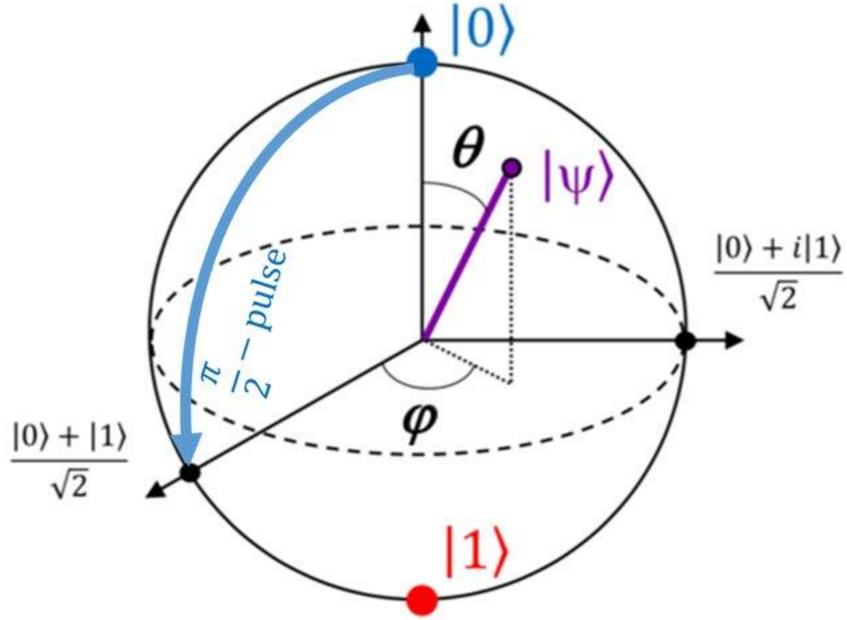
ω_q is close to but not exactly equal to the Josephson plasma frequency ω_p . Treating the lowest two states as a 2-level system, its Hamiltonian can be written like any spin ½ system,

$$H = \frac{\hbar\omega_q}{2}\, \sigma_z \quad , \qquad \sigma_z|0\rangle = |0\rangle \qquad \sigma_z|1\rangle = -1|0\rangle \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Now that we're reduced the qubit to a 2-level system, the dynamics looks very much like NMR for a spin ½ system. Changing the state of the qubit is like rotating the spin so we need some excitation pulse, in this case at the resonant qubit frequency ω_q . A picture of an actual qubit with drive and readout lines is shown below. There are separate microwave drive and readout lines. For readout, the state of the qubit is sensed by its interaction with a microwave resonator, shown by the meander line.



https://enccs.github.io/qas2023/notebooks/Qubit_Spectroscopy_Analysis/



$$|\psi\rangle = cos\frac{\theta}{2}|0\rangle + e^{i\varphi}sin\frac{\theta}{2}|1\rangle$$

https://www.researchgate.net/publication/335028508_A_Review_on_Quantum_Computing_Qubits_Cryogenic_Electronics_and_Cryogenic_MOSFET_Physics/figures?lo=1

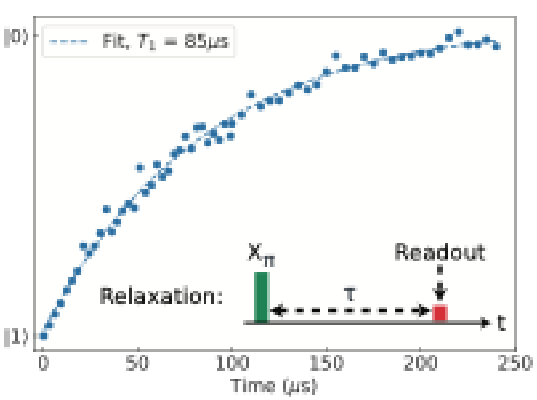
The quantum states for a 2-state system can always be visualized as points on the *Bloch sphere*. This should look to you just like the rotating frame in NMR. For spins with positive gyromagnetic ratio, the higher energy state $|1\rangle$ corresponds to the spin opposite to the B-field direction. The lower energy state $|0\rangle$ would have its spin along the B-field. By applying a burst of energy at the resonant frequency it's possible to do a 90° pulse around the y-axis and rotate the state from,

$$|0\rangle \rightarrow \frac{\pi}{2}\ pulse \rightarrow |\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

This state would have an expectation value of,

$$\langle\psi|\sigma_x|\psi\rangle = 1$$

so it's pointing along the x-direction, as expected. All of the manipulations from NMR such as free-induction decays, inversion recovery, spin echoes are all done on qubits. The data shown is essentially an inversion-recovery measurement on a qubit. A 180° pulse inverts the "spin" and then the qubit state is monitored for varying times afterward until it recovers to its equilibrium value. Here $T_1 = 85\ \mu sec$.



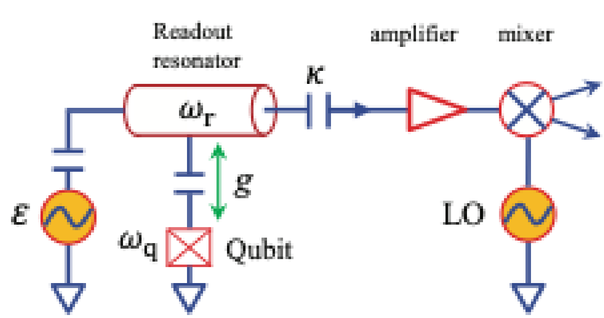
Ideally, to do computations, we'd like T_1 to be infinite but unfortunately the qubit is not entirely isolated from its environment. This problem of *decoherence* is a huge issue for quantum computing. The colder we make the qubit the more of these decoherence pathways are frozen out and the longer it can act like an isolated 2 level system.

"A Quantum Engineer's Guide to Superconducting Qubits, P. Kranz, et. al., arXiv:1904.06560v5, (2021)

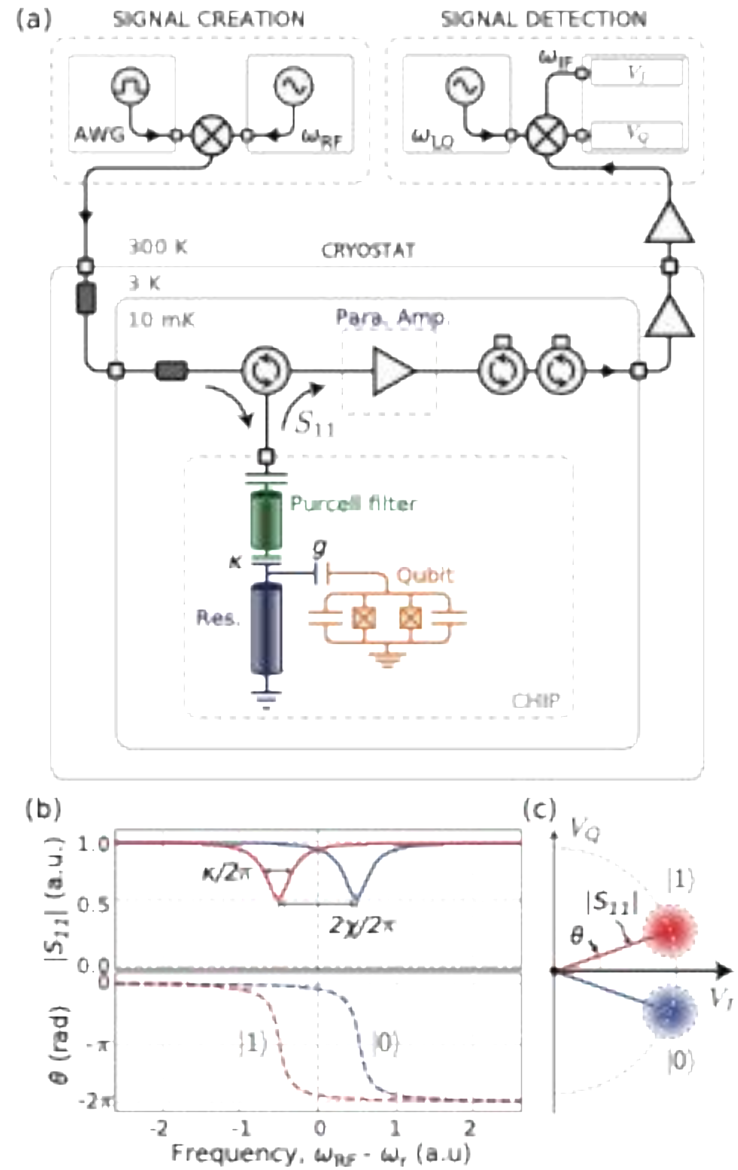
Sensing the qubit state

In NMR it's easy to drive and read out the state of the spins. We apply RF pulses to a coil and generate a time-dependent magnetic field at the resonant frequency. Then we pick up the voltage across the coil that's generated by Faraday's law. The signal is then converted to a lower frequency, digitized and Fourier transformed.

For qubits it's considerably more complicated. To change the state of the qubit it must be driven at or near its resonant frequency ω_q, as in NMR. But to read out the state of the qubit, you don't want to destroy the quantum state it's in. One scheme is to have the qubit coupled to an electromagnetic resonator whose natural frequency ω_r is different from ω_q of the qubit. The frequency of the resonator is changed from ω_r *depending on the state of the qubit*. Now, when a microwave signal is sent to the system (qubit + resonator) the signal reflected back will have a different frequency depending on the state of the qubit. This is called a dispersive readout. It's one type of what they call a *quantum nondemolition* measurement in that the measurement does not destroy the quantum state of the qubit. The basic idea is shown below and the more elaborate circuit diagram is shown on the right. The lower figure show the two different frequencies that are read out, depending on the state of the qubit.



https://arxiv.org/abs/1504.06030v2



"A Quantum Engineer's Guide to Superconducting Qubits, P. Kranz, et. al., arXiv:1904.06560v5, (2021)