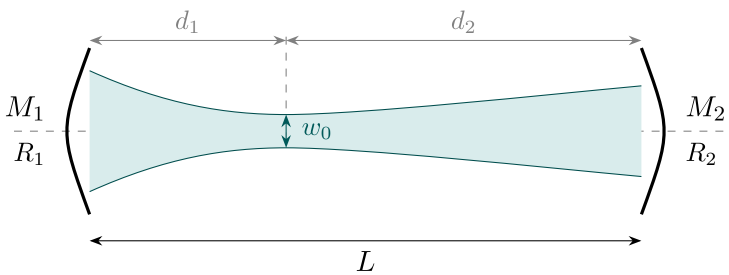
PHYS 525 — Laser Physics

Homework: Gaussian Cavities, CW Output Power, and Semiconductor Lasers

Problem 1: Asymmetric Two-Mirror Resonator



A laser resonator consists of two concave mirrors M_1 and M_2 separated by $L = 20\,\text{cm}$. Mirror M_1 has radius of curvature $R_1 = 50\,\text{cm}$ and mirror M_2 has radius of curvature $R_2 = 40\,\text{cm}$. The wavelength is $\lambda = 1064\,\text{nm}$.

A beam-profile scan along the cavity axis reveals that the intracavity beam waist is located $d_1 = 8\,\text{cm}$ from M_1.

As derived in class, the self-consistent q parameter at mirror M_1 satisfies

$$\frac{1}{q_1} = \frac{D-A}{2B} - \frac{i}{2B}\sqrt{4-(A+D)^2}, \tag{1}$$

where A, B, C, D are elements of the round-trip ray matrix, and

$$\frac{1}{q} = \frac{1}{R_c} - \frac{i\lambda}{\pi w^2}. \tag{2}$$

(a) Round-trip ABCD matrix

Write the round-trip matrix M_{rt}, starting just after reflection from M_1, leaving R_2 symbolic. Use $M_{\text{mirror}}(R) = \left(\begin{smallmatrix} 1 & 0 \\ -2/R & 1 \end{smallmatrix}\right)$ and substitute R_1, R_2, and L numerically. Verify that $\det M_{\text{rt}} = 1$.

(b) Stability

Suppose we want to explore different output coupler choices, i.e. mirrors with different radii of curvature R_2. Express the stability condition $\left|\frac{A+D}{2}\right| < 1$ in terms of R_2 and determine the full range of R_2 for which the cavity remains stable. Verify that the value $R_2 = 40\,\text{cm}$ lies within the stable range.

(c) Verifying the waist location

At the waist, $\mathrm{Re}(1/q) = 0$, which means $A' = D'$ for the round-trip matrix M' starting from the waist plane. Construct M' by conjugating M_{rt} with a free propagation of distance z:

$$M'(z) = M_{\mathrm{prop}}(z)\, M_{\mathrm{rt}}\, M_{\mathrm{prop}}(-z).$$

Show that $A'(z) = A + zC$ and $D'(z) = D - zC$, set $A'(d_1) = D'(d_1)$, and derive the formula $d_1 = (D - A)/2C$. Substitute the numerical values from part (a) and verify that this gives $d_1 = 8\,\mathrm{cm}$, consistent with the beam-profile measurement.

(d) Beam waist radius

Using the numerical matrix elements from part (a) and the waist-plane matrix $M'(d_1)$ from part (c), evaluate the imaginary part of $1/q$ at the waist and hence find w_0.

(e) Physical interpretation

The waist is closer to M_1 than to M_2. In one or two sentences, explain physically why the waist is displaced toward the mirror with the *larger* radius of curvature.

Problem 2: CW Output Power of an Nd:YAG Laser

Consider a linear Nd:YAG laser cavity operating CW at $\lambda = 1064$ nm. Nd:YAG is a *homogeneously broadened* gain medium, meaning that a single lasing mode saturates the gain uniformly across the entire population inversion. The steady-state inversion equations used below apply specifically in this regime.

The two-mirror cavity and gain medium parameters are listed in the table below.

Parameter	Symbol	Value	Units
Stimulated emission cross section	σ	2.8×10^{-19}	cm^2
Upper-state lifetime	τ	230	μs
Back mirror reflectivity	R_1	1.00	—
Output coupler reflectivity	R_2	0.80	—
Single-pass facet transmission	T	0.99	—
Gain medium length	ℓ	5.0	cm
Beam mode radius ($1/e^2$)	w	150	μm
Pump rate density	$\mathcal{R}$	1.13×10^{21}	$\mathrm{cm}^{-3}\,\mathrm{s}^{-1}$

(a) Threshold inversion density.

The gain medium has partially reflective coatings on each facet with single-pass transmission T. Write down the round-trip threshold condition for lasing and solve for the threshold population inversion density ΔN_{th}. Express your answer in cm^{-3}.

(b) Intracavity intensity.

In steady-state CW operation, the population inversion under simultaneous pumping and stimulated emission satisfies

$$\Delta N = \frac{\mathcal{R}\,\tau}{1 + \dfrac{\sigma\tau}{h\nu} I}\,. \tag{3}$$

Using your result from part (a), find the intracavity intensity I_{circ}. What is the physical interpretation of I_{circ}?

(c) Output power.

Using the intracavity intensity from part (b) and the beam mode radius w, calculate the output power P_{out} of the laser. You will need to evaluate the fraction of the intracavity field that exits through the output coupler each round trip.

(d) Optimal output coupling.

For a fixed pump rate $\mathcal{R}$ (and hence fixed small-signal inversion $\Delta N_0 = \mathcal{R}\tau$), determine the output coupler reflectivity R_2^* that maximizes P_{out}.

Note that ΔN_{th} itself depends on R_2 through the round-trip loss condition. You may find R_2^* either:

- *Graphically* — plot P_{out} as a function of R_2 over the range where lasing occurs and read off the maximum; or
- *Analytically* — express $P_{\text{out}}(R_2)$ in closed form, differentiate with respect to R_2, and solve for the optimum. (*Hint:* the substitution $\delta = \ln(1/R_2)$ simplifies the algebra considerably.)

In either case, report the numerical value of R_2^* and the corresponding maximum output power P_{out}^*. How does R_2^* compare to the value $R_2 = 0.80$ used in part (c)?

Problem 3: Threshold Current Density in GaAs Laser Structures

Consider two GaAs-based laser structures: a double heterostructure (DH) and a single quantum well (QW). Assume we are operating the lasers at temperature $T = 0$.

Given parameters:

- $m_e = 9.109 \times 10^{-31}$ kg (free electron mass)
- $m^* = 0.067\, m_e$
- $E_g = 1.42$ eV (bandgap)
- $L_z = 10$ nm (quantum well width)
- $d = 100$ nm (DH active region thickness)
- $\tau = 1$ ns (carrier lifetime, all structures)
- $E_{fc} - E_g = 0.05$ eV (E_{fc} is the quasi-Fermi level in the conduction band)
- $E_n + E_p = 0.04$ eV for the QW (E_n and E_p are quantized electron/hole subbands)

Useful constants:

- $\hbar = 1.055 \times 10^{-34}$ J s
- $e = 1.602 \times 10^{-19}$ C
- 1 eV $= 1.602 \times 10^{-19}$ J

Relevant equations:

The 3D density of states:

$$\rho_{3D}(h\nu) = \frac{1}{2\pi^2}\left(\frac{2m^*}{\hbar^2}\right)^{3/2}(h\nu - E_g)^{1/2} \tag{4}$$

The 2D density of states (per unit volume):

$$\rho_{2D}(h\nu) = \frac{m^*}{\pi\hbar^2 L_z}\,\Theta(h\nu - E_g - E_n - E_p) \tag{5}$$

The Fermi-Dirac distribution:

$$f(E, E_{fc}, T) = \frac{1}{1 + \exp\left(\dfrac{E - E_{fc}}{k_B T}\right)} \tag{6}$$

The threshold current density:

$$J_{th} = \frac{e\, n_{th}\, d}{\tau} \tag{7}$$

(a) Starting from

$$n = \int_0^\infty \rho(h\nu)\, f(h\nu, E_{fc}, T)\, d(h\nu),$$

show that in the limit $T \to 0$ the Fermi-Dirac distribution reduces to a step function, carefully treating the two cases $h\nu < E_{fc}$ and $h\nu > E_{fc}$. Note that ρ_{3D} is zero for $h\nu < E_g$ (the

Bernard-Duraffourg condition: no states exist below the bandgap). Use this to show that the 3D carrier density is:

$$n_{3D} = \frac{1}{3\pi^2}\left(\frac{2m^*}{\hbar^2}\right)^{3/2}(E_{fc} - E_g)^{3/2}.$$

(b) Apply the same $T \to 0$ limit to the 2D case and show that:

$$n_{2D} = \frac{m^*}{\pi\hbar^2 L_z}\left(E_{fc} - E_g - E_n - E_p\right).$$

Explain physically why the step function in ρ_{2D} shifts the lower limit of the integral.

(c) Using the given parameters, calculate n_{3D} and n_{2D} numerically. Then compute J_{th} for each structure using $J_{th} = e\,n\,d/\tau$, noting that $d = L_z$ for the QW. How do the two current densities compare?

(d) Using your results from parts (a) and (b), determine how J_{th} scales with the energy above the respective band edge threshold for each structure. Express your answer in the form $J_{th} \propto \Delta E^{\alpha}$, where ΔE is the appropriate energy above threshold in each case, and identify the exponent α. What does the difference in α between the two structures tell you physically about the advantage of quantum confinement?