489 Spring 2004 Homework 1

Due Wednesday, Jan. 28, 2004

1. Aschroft and Mermin, problem 2.1 (parts a, b, and c).

2. Aschroft and Mermin, problem 2.2 (part a). Note that the derivation does *not* depend upon the free electron approximation; the form of the expression holds for any system of non-interacting fermions if the integral is replaced by a sum over states i with energy ϵ_i.

3. Aschroft and Mermin , problem 2.3 (parts a, b, and c).

4. This problem relates to the change in chemical potential with temperature for a free electron gas in 3D. This is an explicit example of the analysis of A & M in Ch. 2.

   (a) Show that the following integral is obeyed at any temperature:

   $$E_F^{3/2} = \frac{3}{2}\int_0^\infty \frac{E^{1/2}}{exp(\beta(E-\mu))+1} dE \qquad (1)$$

   (b) Show that $\mu = E_F$ at $T = 0$.

   (c) For $k_B T << E_F$, the integral can be approximated by expanding the function $h(E) \equiv E^{1/2}$ around the value at $E = \mu$, $h(E) = h(\mu) + h'(E)_{E=\mu}(E-\mu) + \ldots$. Show that to lowest order in T, the chemical potential is given by the expression (2.78) of A & M:

   $$\mu = E_F\left[1 - \frac{\pi^2}{12}\left(\frac{k_B T}{E_F}\right)^2\right] \qquad (2)$$

   (d) Calculate the change in the chemical potential μ for a typical metal with $E_F = 5eV$ for two cases, $T = 300K$ and $T = 1000K$.

   (e) EXTRA: You do not need to turn this in. Consider the the change in the chemical potential μ for a two-dimensional gas using the result from A & M, problem 2.1 (part c). As discussed in problem 2.1 (part d,e,f) the expansion gives zero to all orders. By doing the analysis more completely, one finds an expression that does not have an analytic expansion around $T = 0$.

5. Interacting electrons. Consider a system with only one single particle state of energy ϵ for either up or down electrons. If no particles are present the energy is defined to be zero. If two electrons are present, the energy is $2\epsilon + U$, where U is the interaction energy.

   (a) List all the possible states for all possible numbers of electrons and give the energy of each state.

   (b) Find expressions for the total number of electrons n and the energy E of the system as a function of T and μ.

(c) Sketch two graphs of n vs. μ for two different values of U: one for the non-interacting case $U = 0$ and one for $U = 2\epsilon$. Assume a fixed $T = 0.1\epsilon$ and consider μ in the range from $\mu < \epsilon$ to $\mu > 2\epsilon + U$ for $T = 0.1\epsilon$ (The graphs do not need to be precise, but they should show the basic features.)