Crystallographic restriction theorem
See: http://en.wikipedia.org/wiki/Crystallographic_restriction_theorem

Problem: Consider rotational symmetry operations of a 2D periodic lattice with the rotation axis perpendicular to the lattice plane. Show that only rotations that are multiples of 60° and 90° can be symmetry operations. Thus, 2, 3, 4, and 6-fold symmetries are allowed, but not 5-fold. This theorem also holds for 3D lattices.

Assume that a 2D lattice is invariant under rotation of angle θ. Let **a** be a primitive lattice vector. Rotate **a** by angle θ to generate **a**', and rotate **a** by angle $-\theta$ to generate **a**". Both **a**' and **a**" must be lattice vectors, and so is **a**' + **a**", which points in either the same or the opposite direction as **a**. Since **a** is primitive (shortest along its direction), **a**' + **a**" = n**a**, where n is an integer (positive, negative, or zero). Taking the dot product of this equation with **a** yields $2a^2 \cos\theta = na^2$. Thus, $\cos\theta = 0, \pm 1/2$, or ± 1. The only possible values of θ are 0°, 60°, 90°, and their multiples.

In 3D, consider an arbitrary lattice vector **R** that is not parallel to the rotation axis. Rotate it by θ to generate **R**'; the difference **R**' - **R** is a nonzero lattice vector perpendicular to the rotation axis. It follows that one can always find a line of lattice points perpendicular to the rotation axis; let **a** be the primitive (shortest) lattice vector for this 1D lattice. The rest of the proof is the same as the 2D case.

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