Lecture 14:

1.) Stark Ladder

We have been considering an electron in an electric field in the TBM. When
\[ \psi(t) = \sum_n C_n(t)|n\rangle, \]
the E.O.M. for the amplitudes is

\[ E C_n = -\mu (C_{n+1} + C_{n-1}) - E_0 n C_n. \]

we showed that
\[ C_1 = e^{\frac{-4iE_o t \sin E_o t}{2k} \cos \left(\frac{E_o t}{2k} + \frac{E_0}{2}\right)} \]

and

\[ \langle n \rangle = 0 \]
\[ \langle n^2 \rangle = \left(\frac{4t \sin E_0 t}{2k}\right)^2 \frac{1}{2n!} \]

This is why the particles do not collect on one side of the lattice.
They never move away from their initial site.

a) Eigenstates

\[ C_{mltl} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i E_0 t \frac{\sin E_0 t}{2}} e^{-i k m} e^{-i \frac{4i E_0 t}{E_0} \sin E_0 t} \cos \frac{k - E_0 t}{2} \, dk. \]

Let \( k \to k + E_0 t/2 \).

\[ \Rightarrow C_{mltl} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i (k + E_0 t) m} e^{-i \frac{4i E_0 t}{E_0} \sin E_0 t} \cos \frac{k - E_0 t}{2} \, dk. \]

Note: There is no change in the limits of integration since we are integrating over the entire unit cell. Here is a useful identity:

\[ e^{i x \cos k} = \sum_{n=-\infty}^{\infty} i^n J_n(x) e^{ink}. \]

\( J_n \) are Bessel functions.

So \[ C_{mltl} = e^{-i E_0 t \frac{\sin E_0 t}{2}} \sum_{n=-\infty}^{\infty} i^n J_n \left( \frac{4i E_0 t}{E_0} \sin E_0 t \right) \int_{-\pi}^{\pi} e^{-i k m} e^{-i \frac{4i E_0 t}{E_0} \sin E_0 t} \cos \frac{k - E_0 t}{2} \, dk. \]

\[ = i^m e^{-i \frac{E_0 mt}{2}} J_m \left( \frac{4i E_0 t}{E_0} \sin E_0 t/2 \right). \]

So we see the solutions involve Bessel functions. Here's the real reason why.
b.) **Starks Ladder**

\[ E_{C_n} = -nE_0C_n - t_0(C_{n+1} + C_{n-1}). \]

In general \[ E = \pm mE_0. \]

\[ \Rightarrow \left[ \frac{(n\pm m)E_0}{-t_0} \right] C_n = C_{n+1} + C_{n-1}. \]

Let's first determine the \( E = 0 \) solution.

\[ \frac{nE_0}{-t_0} C_n = C_{n+1} + C_{n-1}. \]

Recall Bessel's identity: \( \frac{2n}{x} J_n = J_{n+1} + J_{n-1}. \)

Let \( x = -\frac{2t_0}{E_0} \).

\[ \Rightarrow \frac{2n}{x} C_n = C_{n+1} + C_{n-1}. \]

\[ \Rightarrow C_n = J_n \left( x = -\frac{2t_0}{E_0} \right). \]

Let's write the \( E = 0 \) state in the following way

\[ \langle E=0|y|E=0 \rangle = \sum_n C_n \langle E=0|n \rangle \]

\[ = \sum_n J_n (-\frac{2t_0}{E_0}) \langle E=0|n \rangle. \]
What about $\pm E_0$?

\[
(n+1)E_0 C_n = C_{n+1} + C_{n-1}
\]

\[
\frac{2(n+1)E_0}{-2t_o} \rightarrow J_{n+1} + J_n
\]

\[
\rightarrow J_n + J_{n-1}.
\]

\[
\Rightarrow \langle \pm E_0 | \psi(t) \rangle = \sum_n J_{n+1}(\frac{-2t_o}{E_0}) \langle \pm E_0 | n \rangle.
\]

In general

\[
\psi_{\pm m} = \langle \pm m E_0 | \psi(t) \rangle = \sum_n J_{nm}(\frac{-2t_o}{E_0}) \langle \pm m E_0 | n \rangle.
\]

The state with energy $\pm m$ has its maximum amplitude on site $n \pm m = 0$ or site $\pm m$. The state with energy $E_0$ is peaked at site $-1$.

$\Rightarrow J_{\pm m}$ is the amplitude that a particle is on site $n$ with energy $\pm m$. The $\psi_{\pm m}$ are the Static Ladder states. Note there is an $E=0$ state because the potential is unbounded. $\Rightarrow$ there is no zero point energy. $\Delta x = \infty$ and $\Delta p = 0$.

2. Momentum Relaxation

To get long-range transport, one needs to include some sort of mechanism for
relaxation of the momentum.

\[ \dot{p} = -eE - \frac{m}{\epsilon} \dot{r} \quad \text{and} \quad \dot{r} = \frac{\partial \Sigma}{\partial \epsilon} \]

dissipation mechanism.

You will solve this as a HW problem.

One word on the convention for particles and holes:

\[ \Sigma_k = \Sigma_0 - 2t \cos ka \]

\[ V_k = \frac{2t \alpha \sin ka}{\epsilon} \]

\[ \frac{1}{m} = \frac{1}{\epsilon^2} \frac{2 \Sigma}{\partial \epsilon^2} = \frac{2t \alpha^2}{\epsilon^2} \cos ka \]

\[ \cos ka \geq 0 \quad k \in [0, \pi/2a] \quad M \geq 0 \]

\[ \cos ka \leq 0 \quad k \in [\pi/2a, \pi/a] \quad M \leq 0 \]