1. This problem concerns the band structure of a single sheet of carbon, graphene (see Fig. (1)). Nearest neighbour atoms are connected by the set of vectors,
\[ \delta_1 = \frac{a}{2} \left( 1, \sqrt{3} \right), \quad \delta_2 = \frac{a}{2} \left( 1, -\sqrt{3} \right), \quad \delta_3 = -a \left( 1, 0 \right) \]
and the primitive lattice vectors are
\[ a_1 = \frac{a}{2} \left( 3, \sqrt{3} \right), \quad a_2 = \frac{a}{2} \left( 3, -\sqrt{3} \right) \]
where \( a = 1.42\,\text{Å} \), the nearest-neighbour carbon-carbon spacing. The first Brillouin zone shown in Fig. (1c) is spanned by the reciprocal vectors,
\[ b_1 = \frac{2\pi}{3a} \left( 1, \sqrt{3} \right), \quad b_2 = \frac{2\pi}{3a} \left( 1, -\sqrt{3} \right). \]
The corners of the first Brillouin zone are located at
\[ K = \frac{2\pi}{3a} \left( 1, \frac{1}{\sqrt{3}} \right), \quad K' = \frac{2\pi}{3a} \left( 1, -\frac{1}{\sqrt{3}} \right). \]
You are to a) write down the tight-binding Hamiltonian for this system and solve it for the energy bands, b) determine the values (there are two) of \( k \) for which the energy dispersion vanishes, c) show that the dispersion can be written as
\[ E(q) = \pm \hbar v_F |q| \]
where \( v_F = 3t/2ha \approx 10^6 \text{m/s} \) and \( q \) is the deviation from the value of the momentum at the zero crossing, and d) show that the effective Hamiltonian near the zero crossings can be written as
\[ H \equiv \hbar v_F \begin{pmatrix} 0 & q_x + iq_y \\ -iq_y & 0 \end{pmatrix} = \hbar v_F \sigma \cdot q \]
for the \( K' \) point and
\[ H \equiv \hbar v_F \begin{pmatrix} 0 & -q_x - iq_y \\ q_x - iq_y & 0 \end{pmatrix} = \hbar v_F (-q_x \sigma_x + q_y \sigma_y) \]
for the \( K \) point.
FIG. 1. Graphene lattice. a) Two interpenetrating honeycomb lattices describing the structure of graphene. The star of David shows the penetrating sublattices. b) Primitive lattice vectors and, $\mathbf{a}_i$, and nearest-neighbour lattice vectors, $\delta_i$. The labels A and B refer to the two kinds of carbon atoms in graphene. c) First Brillouin zone spanned by the lattice vectors $\mathbf{b}_i$. 