Show explicitly that
\[
\int_0^{p_F} \frac{dp}{(2\pi\hbar)^3} \epsilon_{\text{exch}}(p) = -\frac{e^2 p_F^4}{4\pi^3 \hbar^4}
\] (1)

Consider a uniform electron gas that interacts via a potential of the form \( V(r) = V_0 e^{-r/a} / r \). a) Solve the Hartree-Fock equations for this system for the eigenfunctions and excitation spectrum, \( \epsilon(p) \). Evaluate the Fermi energy \( \epsilon_F = \mu \). b) At the Hartree-Fock level, show that the effective mass \( m^* \) is determined solely by the exchange contribution. Compute explicitly \( m^* \) in the limits \( k_F a \ll 1 \) and \( k_F a \gg 1 \). c) Show that the exchange interaction contribution to \( \epsilon_F \) is negligible when \( k_F a \gg 1 \) and that the direct and exchange terms are comparable for a short-range interaction with \( k_F a \ll 1 \).

Calculate the density of states for the tight-binding model on a square lattice. You are to assume that only the nearest-neighbour matrix element is non-zero.