Assigned Thursday, October 6, 2005. Due Thursday, October 20, 2005

1. Excitations of electrons in condensed matter

Describe in words (using simple figures if you wish) the following:

a. Reasons why we expect the imaginary part of the self-energy of an electron to vary as $(E - E_F)^2$ for energy E near the Fermi energy E_F, based upon perturbation theory in the electron-electron interactions.

b. Why we consider a plasmon to be a "collective excitation" of all the electrons, whereas a quasiparticle is a "single particle excitation" affected by interactions with all the electrons.

2. Instability of a Fermi Liquid to Magnetism

In class notes we defined Fermi liquid parameters by
$E = E_0 + \sum_{k,\sigma} \epsilon(k,\sigma)\delta n(k,\sigma) + \frac{1}{2}\sum_{k,\sigma,k',\sigma'} f(k,\sigma,k',\sigma')\delta n(k,\sigma)\delta n(k',\sigma') + \ldots$
and dimensionless parameters by F_s and F_a for the symmetric (average over spins) and antisymmetric (difference between spins):
$F_s(\mathbf{k}\cdot\mathbf{k}') = \rho(E_F) f_s(\mathbf{k}\cdot\mathbf{k}') = \sum_l F_{s,l} P_l(cos(\theta)$, and F_a = similar expression. Here $\rho(E_F)$ is the density of single particle states at the Fermi energy, which is governed by the effective mass in Fermi Liquid Theory.

a) Based on physical reasoning give reasons why one might expect the interaction between like spins in a Fermi liquid to be more attractive (less repulsive) than for opposite spins. If this happens then the Fermi liquid parameter f_0^a or F_0^a would be negative.

b) Assuming that there is such an effect, find the quantitative expression for the instability of the Fermion system to ferromagnetism. Give your expressions in terms of the Fermi liquid parameter f_0^a and the density of states at the Fermi energy.

c) Consider the instability in the case of three dimensional Jellium. There one expects the Fermi liquid interaction parameter to scale as $\frac{1}{volume} * \frac{1}{r_s}$, where the last factor is the expected scaling for any Coulomb interaction. What is the expected scaling for the density of states with r_s in Jellium. From this, what can you conclude about the expected instability to ferromagnetism in the high density limit, and in the low density limit?

3. Electron-Phonon Interaction

The object of this problem is to show the general features of the mass enhancement of electrons near the Fermi energy by interaction with phonons. You do NOT need to work out the expressions for the self-energy. All that is needed is to consider the form of the self-energy and to derive the real part from reasonable expressions for the imaginary part. It is convenient to use the *retarded* form for the self-energy, which allows the use of the Kramers-Kronig (KK) relations. Also it is convenient to choose the Fermi energy $E_F = 0$.

a. For electron energies $|E| >> \omega_{ph}$, where ω_{ph} is a typical phonon frequency, we expect $Im\Sigma_k(E)$ is roughly constant and we can set $Im\Sigma_k(E) = \frac{1}{\tau}$. Argue in words or use the "golden rule" expression for the imaginary part of the electron self energy, that this is a reasonable approximation.

b. However, $Im\Sigma_k(E)$ must $\rightarrow 0$ as $E \rightarrow 0$. Let us approximate this by:
$Im\Sigma_k(E) = \frac{1}{\tau}$, $|E| > \omega_{ph}$,
$Im\Sigma_k(E) = \frac{1}{\tau}\frac{E^2}{\omega_{ph}^2}$, $|E| < \omega_{ph}$

Show using the KK relations, show that this leads to $Im\Sigma_k(E)$ that varies linearly with E near $E = 0$ and give the magnitude of the slope in terms of $\frac{1}{\tau}$ and ω_{ph}. Give also the mass enhancement factor $\lambda \equiv -\frac{d\Sigma}{dE}$. [Hint: This may seem tricky since the integral in the KK transform is over all E and the integrals do not appear to converge. This is not a problem because a constant $Im\Sigma_k(E) = \frac{1}{\tau}$ would give a constant $Re\Sigma_k(E)$ with $\frac{d\Sigma}{dE} = 0$ at $E = 0$. Thus the slope is independent of a constant value of $Im\Sigma_k(E)$ and desired answer can be found from the KK transform of only the E dependent part of $Im\Sigma_k(E)$ near the Fermi energy.]

c. Finally give a give rough estimate of the magnitude of λ if we assume a reasonable (large) value of $\frac{1}{\tau}$ to be of order the phonon frequency ω_{ph}.

4. Anderson Impurity Model in the Hartree-Fock Approximation The object of this problem is to carry out the Hartree-Fock approximate solution of the Anderson Model for impurities in a metal. The primary result will be to find the ranges of parameters where the solution is a) a singlet state with no magnetic moment and b) a degenerate pair of states each with a moment. The letter case is called the "local moment regime". The best reference is the original paper (P. W. Anderson, Phys. Rev. 124, 41 (1961).)

The Hamiltonian is given by

$$H = \sum_{\sigma} \epsilon_L c^{+}_{L,\sigma} c_{L,\sigma} + U n_{L,\uparrow} n_{L,\downarrow} + \sum_{k,\sigma} \epsilon_k c^{+}_{k,\sigma} c_{k,\sigma} + V \sum_{k,\sigma} \left(c^{+}_{k,\sigma} c_{L,\sigma} + h.c. \right) \tag{1}$$

For the non-interacting ($U = 0$) case you may may use the form of the Green's function derived in the class notes for a "flat" density of states

$$G_{\sigma}(L, L, \omega) = \frac{1}{\omega - \epsilon_L - i\pi\Delta} \tag{2}$$

where Δ = constant and ϵ_L is assumed to include any shift from the real part of the self-energy.

In the H-F approximation, the Greens function for each spin is given by the above equation with

$$\epsilon_L \to \epsilon_{L,\sigma} = \epsilon_L + U\langle n_{L,-\sigma} \rangle. \tag{3}$$

a) Show that in the H-F approximation the number of localized electrons of each spin type is given by

$$n_{L,\sigma} = \int_{-\infty}^{E_F} dE \rho_{L,\sigma}(E) = \frac{1}{\pi} cot^{-1} \left(\frac{\epsilon_L + U\langle n_{L,-\sigma} - E_F}{\Delta} \right) \tag{4}$$

b) In terms of the variables $x = \frac{E_F - \epsilon_L}{U}$, $y = \frac{U}{\Delta}$, $n_1 = n_{L\uparrow}$ and $n_2 = n_{L\downarrow}$, show that the equations can be written:

$cot n_1 - y(n_2 - x) = 0$

$cot \pi n_2 - y(n_1 - x) = 0$.

c) Show graphically that for some values of x and y there is only one non-magnetic solution with $n_1 = n_2$, whereas for other values of x and y, there are three solutions.

Show this by graphing n_2 as a function of n_1 from the first equation and n_1 as a function of n_2 from the second equation. Plot the solutions for n_2 vs. n_1 and show examples of the two types of solutions.

d) Show that the boundary between the non-magnetic and magnetic regimes is given by the condition that the two curves have the same slope at the point $n_1 = n_2$, and that this leads to the relation of critical values of n_c and y_c given by
$\frac{\pi}{sin^2 \pi n_c} = y_c$.
e) Show qualitatively that the boundary of the magnetic regime has the form found by Anderson.