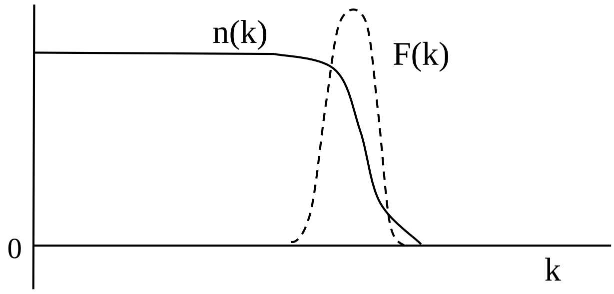

Figure 1: Figure for Problem 1. Schematic illustration of the momentum and anomalous condensate pair amplitude function in the BCS theory.

Assigned Tuesday, November 17, 2005. Due Friday, December 5, 2005

1. Momentum distribution n_k and condensation amplitude F_k in the BCS state

   A. Derive an explicit expression for the momentum distribution $n_k = \langle\tilde{\phi}|a^+_{k\sigma}a_{k\sigma}|\tilde{\phi}\rangle$, where $\tilde{\phi}$ is the BCS state. Show that $n_k = v(k)^2$ from which it follows that the expression for n_k can be written in terms of the unperturbed electron states ϵ^0_k and the gap function Δ_0 which is taken be be constant independent of k for simplicity. Show the distribution has the continuous form shown in the figure for the usual case $\Delta_0 << E_F$.

   B. Similarly show that the anomalous expectation value $F_k\langle\tilde{\phi}|a^+_{k\uparrow}a^+_{-k\downarrow}|\tilde{\phi}\rangle = U(k)v(k)$ and thus has a peak near the Fermi energy as shown in the figure. The anomalous expectation value F_k is the amplitude of state k in the condensate.

2. Calculate the mean square radius $\langle r^2\rangle^{1/2}$ for a Cooper pair in the singlet state. Assume an isotropic Fermi surface and constant interaction V independent of $\mathbf{k}$ for the energy range ω_D. You may assume the bound state energy is $E << E_F$. [Hint: it may be helpful to use the expressions in $\mathbf{k}$ space and use $\mathbf{r} \rightarrow d/d\mathbf{k}$.] Note: This problem is essentially the same as Phillips problem 11.5.

3. Bogoliubov transformation

   A. First go through the steps shown in the notes to write the hamiltonian for the system with BCS pairing as (Note that there is a mistake in the expression for H_M in typed notes show passed out in class)
   $H_M = \sum_{k\sigma} \epsilon^0_k a^+_{k\sigma}a_{k\sigma} - \sum_k(\Delta_k a^+_{k\uparrow}a^+_{-k\downarrow} \ + \ c.c. \ - \ \Delta_k b^*_k)$
   This shows that the gap Δ play the role of a mean "pairing" field analogous to an ordinary effective potential that is a mean field that acts on the density.

   B. The Bogoliubov transformation expresses the electron creation and annihilation operators as linear combinations pair breaking operators
   $a_{k\uparrow} = u^*_k\gamma_{k0} + v_k\gamma^+_{k1}$
   $a^+_{-k\downarrow} = v^*_k\gamma_{k0} + u_k\gamma^+_{k1}$
   and similarly for $a^+_{k\uparrow}$ and $a_{-k\downarrow}$.

Show that the γ operators obey fermion anticomutation rules.

C. Show that the transformation leads to an expression for hamiltonian
$H = E_0 + \sum_{k\sigma} \epsilon_k (\gamma_{k0}^{+} \gamma_{k0} + \gamma_{k1}^{+} \gamma_{k1})$
where E_0 is the ground state energy that includes the BCS binding energy and the excitations are non-interacting fermions with energy
$\epsilon_k = \left[(\epsilon_k^0)^2 + \Delta_k^2\right]^{-1/2}$
This is the energy to add an unpaired fermion which has a gap.

4. Using the Ginsburg-Landau approach show that the free energy of a superconductor relative to the normal state is given by
$F_S - F_N = -\frac{1}{8\pi} H_c^2$
where F_S and F_N are the free energies evaluated at zero magnetic field and any given temperature $T < T_c$, and H_c is the critical magnetic field at that temperature.

   This shows that the free energy of the superconducting state is determined directly from measured quantities and was established before a successful microscopic theory was formulated. (This problem is the same as Phillips problem 11.1.)