561 Fall 2005 Lecture 7

Self-energies and Quasiparticles in Many Body Perturbation Theory (T= 0 Formalism)

Following Mahan Ch. 2, 3.3, 3.4; Other discussions in Fetter Ch. 3,4, Abrikosov, ...

1. Dyson's Equation and Proper Self Energies $\Sigma^\star$

From the previous notes, the exact Green's function can always be expressed in the form:

$$G_\lambda(E) = G^0_\lambda(E) + G^0_\lambda(E)\Sigma(E)_\lambda G^0_\lambda(E) \equiv G^0_\lambda(E) + G^0_\lambda(E)\Sigma^\star_\lambda(E)G_\lambda(E), \tag{1}$$

where $\Sigma^\star_\lambda(E)$ is the proper self-energy. (As defined in Fetter and the previous notes, $\Sigma^\star$ is the simpler "irreducible" set of diagrams that cannot be divided into two parts by cutting a single G^0 line). This holds for any of the Green's functions considered here: retarded, time-ordered, phonon, electron, ...

From the expression above for G^0, it follows that G can always be written as

$$G_\lambda(E) = \frac{1}{G^0_\lambda(E)^{-1} - \Sigma^\star_\lambda(E)}, \tag{2}$$

where $\Sigma^\star_\lambda(E)$ is defined for complex values of the energy E. Thus $Im\Sigma^\star_\lambda(E)$ must be defined properly so that the Green's function has the proper analytic structure for $G^{time-ordered}$, $G^{retarded}$, etc. This means that $\Sigma^\star_\lambda(E)$ also is time-ordered, retarded, etc.

For a non-interacting hamiltonian H^0 with eigenvalues ϵ^0 (where λ labels the quantum numbers such as momentum, spin, . . .), this can be written

$$G_\lambda(E) = \frac{1}{E - \epsilon^0_\lambda - \Sigma^\star_\lambda(E)}, \tag{3}$$

This holds for example for electrons with H^0 a appropriately chosen non-interacting hamiltonian; similar expressions for phonons and any case in which G^0 is a Green's function for a non-interacting hamiltonian. Thus the proper self energy has the interpretation of the modification of the independent particle energy due to the effects of the interactions. *This is the most useful quantity for our studies.*

The equation above can be written

$$\Sigma^\star_\lambda(E) = G^0_\lambda(E)^{-1} - G_\lambda(E)^{-1}, \tag{4}$$

and it follows that the form of $\Sigma^\star_\lambda(E)$ for E in the complex plane is closely related to our previous discussion of response functions and Green's functions.

2. Quasiparticles

The energy $\epsilon^0_\lambda + \Sigma^\star_\lambda(E)$ can be considered to be the (complex) energy of the *quasiparticle* with quantum numbers λ. Note we are making the ansatz that the true solutions have the same quantum numbers as the non-interacting states - i.e. there is a one-to-one mapping. This is equivalent to the assumption that there is no symmetry breaking (discussed later in course) so that conservation laws (e.g., momentum, spin, ...) are rigorously maintained. (See article in "Reference Frame" by P. W. Anderson, p. 11, Physics Today, February, 2005, "Brainwashed by Feynman" who tries to make the case that this is misleading and the series may not sum as you think!)

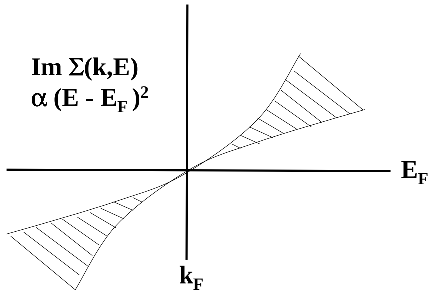

Figure 1: Schematic illustration of the peak in the Green's function, with width $\propto Im\Sigma^\star_\lambda(E)$, which vanishes at the Fermi energy and increases as the energy is changed from the Fermi energy, either higher (for adding electrons above E_F) or lower (from removing electrons below E_F.

$Re\Sigma^\star_\lambda(E)$: The real part of $\Sigma^\star_\lambda(E)$ is a shift in the energies of the quasiparticles. This gives renormalized energies for the quasiparticles.

$Im\Sigma^\star_\lambda(E)$: The imaginary part is a lifetime of the quasiparticles. To see this we go to the continuum limit where the imaginary part is the decay into the continuum. Depending upon the desired Green's function, $Im\Sigma^\star_\lambda(E)$ can have retarded form, time-ordered, etc. The analytic properties in each case follow from our previous definitions.

If the renormalized Green's function still has sharp, well-defined peaks (which means that the lifetime broadened width, i.e., the imaginary part $Im\Sigma^\star_\lambda(E)$ is small compared to the real part $\epsilon^0_\lambda + Re\Sigma^\star_\lambda(E)$ then one can still think of the excitations as well-defined quasiparticles.

Example: Electrons near the Fermi energy.
The zero of energy for the real part real part $\epsilon^0_\lambda + Re\Sigma^\star_\lambda(E)$ is the Fermi energy of the true interacting system. This is because at $T = 0$ the Fermi energy μ is defined to be the lowest energy at which an electron can be added, and the highest at which an electron can be removed. This is a well-defined many-body concept.

Then the criterion that electron quasiparticles are well defined is that the imaginary part of the self energy $Im\Sigma^\star_\lambda(E)$ go to zero faster than the real part of the quasiparticle energy $\epsilon^0_\lambda + Re\Sigma^\star_\lambda(E) - \mu$, as $E \to \mu$.

See figure for the expected behavior - more later on why it is expected to have $Im\Sigma^\star_\lambda(E) \propto (E - E_F)^2$. **3. Renormalization as function of momentum**

In a crystal (or the homogeneous gas) in which the states are labelled by momentum k, the dispersion of the quasiparticles is "renormalized"

$$\frac{d\epsilon_k}{dk} = \frac{d\epsilon^0_k}{dk} + \frac{dRe\Sigma^\star_k}{dk}. \tag{5}$$

This is the important effect in the Hartree-Fock approximation; however, in general this is not the full renormalization.

4. Renormalization as function of energy
See Mahan section 3.3, 5.8A, and 6.4B)

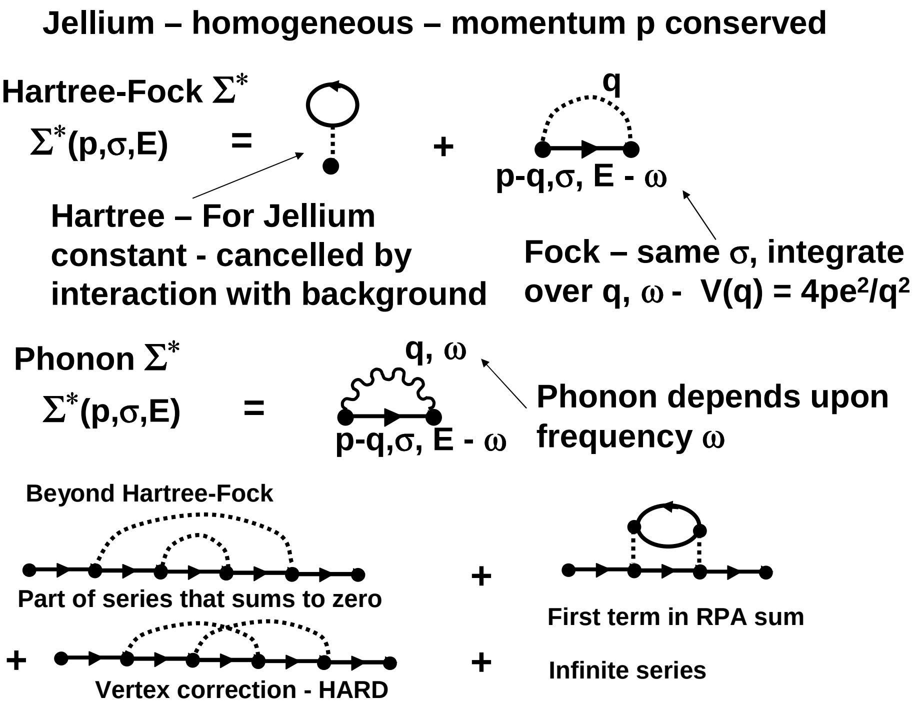

Figure 2: Examples of Diagrams.

For E near the QP energy $\epsilon_\lambda = \epsilon_\lambda^0 + Re\Sigma_\lambda^\star(\epsilon_\lambda)$, one can expand $\Sigma(E)$ to find

$$Re\Sigma_\lambda^\star(E) = Re\Sigma_\lambda^\star(\epsilon_\lambda) + (E - \epsilon_\lambda)\left[\frac{d\ Re\Sigma_\lambda^\star(E)}{dE}\right]_{E=\epsilon_\lambda}, \tag{6}$$

which leads to

$$G_\lambda(E) = \frac{1}{E - \epsilon_\lambda^0 - \Sigma_\lambda^\star(E)} \approx \frac{Z_\lambda}{E - \epsilon_\lambda - Z_\lambda Im\Sigma_\lambda^\star(\epsilon_\lambda)}, \tag{7}$$

which is exactly like an ordinary G^0 except that the energies are renormalized and the weight Z_λ is renormalized. (One can show that the weight $Z < 1$ in physically realistic cases.)

5. Properties of $G_\lambda(E)$

Weighted density of states related to retarded form:

$$\rho_{total}(E) = -\frac{1}{\pi}\sum_\lambda ImG_\lambda^{ret}(E) \ \ with \ \ \rho_\lambda(E) = -\frac{1}{\pi}ImG_\lambda^{ret}(E) \tag{8}$$

6. Examples of Diagrams

See Figure.