561 Fall 2005 Lecture 19

Superconductivity: Microscopic Theory of the Condensed State - BCS Theory

References:

Phillips, Ch 11; Tinkham, "Intro. to Superconductivity", Ch 2; de Gennes, "Superconductivity of Metals and Alloys, Ch 4; Brief summaries in Aschroft and Mermin and Kittel, "Intro. to SSP"; Green's function descriptions in Mahan Ch 9 (second Ed.) Ch 10 (third Ed.)

1. Cooper pairs

- Attractive interaction always leads to instability of Fermi sea to pairing
- Model Potential $V_{k,k'} = -V$ for ϵ^0_k and $\epsilon^0_{k'}$ within an energy ω_D of E_F, $V = 0$ otherwise.
- A pair of electrons (or a pair of holes) of zero total momentum interacts to form a bound state with binding energy $= 2\omega_D exp(-1/N(0)V)$.
  The solution for the energy and pair wavefunction can be derived readily, as done in de Gennes Sec. 4.1 and Phillips Sec. 11.6.1. We will follow this approach. It is also equivalent to a summation of "ladder" diagrams s discuss by Mahan.
- Solution for model potential $V_{k,k'} = -V/\Omega$ for $\frac{1}{2}k^2 - E_F < \omega_D$, $\frac{1}{2}k'^2 - E_F < \omega_D$, and $V_{k,k'} = 0$ otherwise. Here Ω is the volume.
- Leads to instability of Fermi surface at T=0 for any attractive interaction, no matter how small.
- Note that equations are essentially the same as for the Peierls instability, but in this case the instability is in any dimension. Note also the similarity to the Kondo effect.
- Finite temperature broadens Fermi distribution and decreases the effect - pairs unbind at characteristic temperature, qualitatively the same as in the full BCS theory (see notes below).

2. Attractive interaction due to phonons

- Phonons and phonon Green's function described in Lecture 9. There we derived that phonons cause large effects upon the self energy of electrons with energies $|E-E_F| <\approx \omega_D$ where ω_D is a typical phonon energy, the Debye energy.
  Similarly the electrons cause large effects upon the phonon energies.
- Induced electron-electron interactions

  The same interactions cause scattering of electrons mediated by phonons, which leads to an attractive interaction electrons with energies $|E - E_F| <\approx \omega_D$. The interaction is just the diagram shown in Fig. 1 which involves the phonon Green's function. The expression for the effective electron-electron interaction is a sum over all phonons

$$H_{eff} = \sum_{q,p,p'} \frac{|V_q|^2}{\omega^2 - \omega_q^2 + i\eta} c^+_{p+q} c^+_{p'-q} c_p c_{p'}, \tag{1}$$

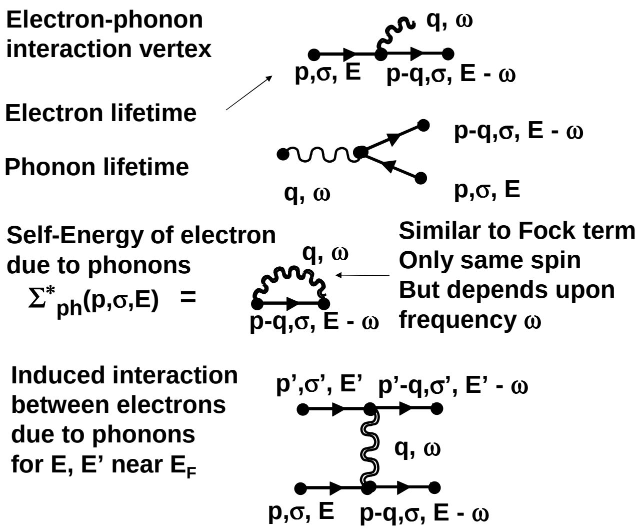

Figure 1: Diagram for the effective interaction shown at the bottom where the double wavy line is the phonon Green's function. This symbols and the upper diagrams are from Lecture 9.

where ω is the energy transfer between the electrons. This is described in class and is given in more detail by Nozieres and Pines, vol. 1, p 243 ff. The same expression is given in Phillips, Eq. (11.35) derived from an approach using a similarity transformation.

- Dynamical frequency dependent effect - i.e., a retarded interaction
- Any spin-independent boson could serve to give attraction

3. Condensed BCS pair state

- Mean field theory for pairs influenced only by the average effects of other pairs. First we describe the coherent ground state of cooper pairs following de Gennes, 4-3.
- BCS introduced wave function that does not have definite particle number. This greatly simplifies the theory
$\tilde{\phi} = \Pi_k(u_k + v_k a^+_{k\uparrow} a^+_{-k\downarrow})|0\rangle$ For a large number of electrons N the fluctuations are of order $N^{-1/2}$.
- The fact that the BCS function is a state in which the wavefunction has a fixed phase and not a fixed number is fundamental to the superconducting state. Instead of fixed number there is a well-defined order parameter Which is the gap function Δ. More about this in the following lecture.
- Same as usual non-interacting Fermi sea if $u_k = 0$ and $v_k = 1$ for $k < k_F$. Interesting effects come from the coherent state with $u_k \neq 0$ and $v_k \neq 0$ for the same k. Note that then one must have $v_k \neq 0$ for $k > k_F$.

- Self-consistent solution for pair function gives a gap in the spectrum, i.e., a condensed bound state of the entire system
  $\Delta_k = -\sum_l V_{kl} u_l v_l = -\sum_l V_{kl} \frac{\Delta_l}{2((\epsilon_l^0)^2 - \Delta_l^2)^{1/2}}$

- Solution for the case of $\Delta_k = \Delta$ is either:
  $\Delta = 0$ (normal metal) or $\Delta = 2\omega_D e^{\frac{1}{N(0)V}}$

- Distribution of occupation of k states. Normal amplitude $n_k = v_k^2$. "Condensate amplitude" $f_k = u_k v_k$. See schematic figures in de Gennes and in the handwritten notes.

4. Finite temperature calculations

- The only change from the T=0 expressions is to include Fermi factors f in the self-consistent solution for the gap.
  $\Delta_k = -\sum_l V_{kl} u_l v_l = -\sum_l V_{kl} \frac{\Delta_l}{2((\epsilon_l^0)^2 - \Delta_l^2)^{1/2}} [1 - 2f(((\epsilon_l^0)^2 - \Delta_l^2)^{1/2})]$

- Truncating Green's functions equations by factorizing in a mean field fashion (exactly the same effect as the BCS mean field theory)

- Finite temperature expressions

5. Canonical (Bogoliubov) Transformation to give ground and excited states

- Transformation that gives the same solution as the BCS theory for the ground state, but is easier for calculation of excited states, inclusion of inhomogeneity, etc.

- The superconducting state is a condensed state with a non-zero pair expectation value $b_k = \langle a_{k\uparrow} a_{-k\downarrow} \rangle$, which is zero in the normal state, non-zero in the condensed state. In terms of the anomalous b_k, the gap is given by
  $\Delta_k = -\sum_l V_{kl} b_l$
  Thus the gap is also an order parameter.

- Pair hamiltonian
  $H_M = \sum_{k\sigma} \epsilon_k^0 a_{k\sigma}^+ a_{k\sigma} - \sum_k (\Delta_k a_{k\uparrow}^+ a_{-k\downarrow}^+ \ + \ c.c. \ - \ \Delta_k b_k^*)$

- The Bogoliubov transformation expresses the electron creation and annihilation operators as linear combinations pair breaking operators
  $a_{k\uparrow} = u_k^* \gamma_{k0} + v_k \gamma_{k1}^+$
  $a_{-k\downarrow}^+ = v_k^* \gamma_{k0} + u_k \gamma_{k1}^+$
  and similarly for $a_{k\uparrow}^+$ and $a_{-k\downarrow}$.
  The result is a hamiltonian
  $H_M = E_0 + \sum_{k\sigma} \epsilon_k (\gamma_{k0}^+ \gamma_{k0} + \gamma_{k1}^+ \gamma_{k1})$
  where E_0 is the ground state energy that includes the BCS binding energy and the excitations are non-interacting fermions with energy
  $\epsilon(k) = \left[(\epsilon^0(k))^2 + \Delta(k)^2\right]^{-1/2}$
  This is the energy to add an unpaired fermion which has a gap.

- Gap for adding or subtracting single electrons Experiments: tunnelling, photoemission

  See figure of photoemission data on high T_c superconductors. The first time there has been resolution to observe the gap directly in photoemission. In this case the gap is "d-wave", i.e., the gap function $\Delta(k)$ has zeros and changes sign as a function of position on the fermi surface.

6. Thermal properties

- In the mean field BCS, T enters as an average reduction of the pair strength. Self consistent solution gives reduction for $T < T_c$ and $\Delta = 0$ (normal state) for $T > T_c$.
- Specific Heat Jump at transition.
- Figure of mean field $\Delta(T)$ with the famous relation $\Delta(0) = 1.76 k_B T_c$

7. Description in terms of Greens functions - Mahan 9.2 (10.2 in 3rd Ed.) - Not covered here

- Ordinary and anomalous Green's functions
- Truncating Green's functions equations by factorizing in a mean field fashion (exactly the same effect as the BCS mean field theory)
- Finite temperature expressions