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Quantum Hall Effect(s)

Phillips Ch. 14 gives a nice discussion and many references.
There are many excellent papers including many original papers that are very readable:
Original explanation of integer QHE: R. B. Laughlin, Phys. Rev. B 23, 5632 (1981)
Short reviews of the fractional QHE (with references and comparisons to the integer QHE) in the Nobel prize lectures: R. B. Laughlin, Rev. Mod. Phys. 71, 863 (1998); H. L. Stormer, Rev. Mod. Phys. 71, 875 (1998); D. C. Tsui, Rev. Mod. Phys. 71, 891 (1998).
Very recent description of fractional charge excitations in Physics Today: J. K. Jain, Physics Today, April, 2005, p. 39.
Collections: A. H. MacDonald, ed., "Quantum Hall Effect: A Perspective" (Kluwer, 1990); R. E. Prange and S. M. Girvin, "The Quantum Hall Effect" second edition (Faller, 1990).

1. Hall Effect

See Phillips Fig. 14.1; also, for example, Kittel, p 148

Consider a system of particles with charge q and density ρ. When a current $J_x = \rho q v_x$ flows in a conductor in the geometry shown, there must be a transverse electric field to balance the force due to the magnetic field, $qE_y = (1/c)qv_xB_z = (1/\rho)J_xB_z$. Also $J_x = \sigma_{xx}E_x = (\rho q^2\tau/m)E_x$ and the cyclotron frequency is given by $\omega_c = qB/(mc)$. Putting this together gives the transverse Hall conductance

$$\sigma_{xy} \equiv \frac{J_x}{E_y} = -\frac{\rho q c}{B_z} \tag{1}$$

Note: This is independent of electron-electron interactions! The electrons move together without changing their relative positions.

2. Ordinary Hall effect in two dimensions

The same expressions apply. The important effects are due to the quantum mechanics of charged particles in two dimensions.

In zero magnetic field the density of states on a 2d electron gas is constant for $E > E_{min}$. In a field the levels split into delta functions called Landau levels, each with degenerate states.

Straightforward to determine the number of degenerate states per unit area in each Landau level. Two ways: using simple counting arguments and the level splitting; using the size and spacing of the orbits in the solution for the states in a particular gauge.

See Phillips Fig. 14.3.

3. Integer Quantum Hall Effect

- We assume the magnetic fields are so large that only one spin of electrons is present. (This is not obvious! One does not expect the spin splitting to be so large, but it is in a semiconductor like GaAs. This is due to electron interactions - a simple effect that can be taken into account phenomenologically.)

- Exact solutions for non-interacting electrons in homogeneous system. The key feature of two dimensions is that the eigenfunctions are Landau orbits with quantized energies $E_n = (n+1/2)\hbar\omega_c$, $n = 1, \ldots$. The wavefunctions are gaussians (in the Landau gauge)

$$\psi_{k,1}(x,y) \propto e^{ikx} e^{(y+y_0-k)^2/2} \tag{2}$$

where $y_0 = qEl/(\hbar\omega_c)$ with the characteristic length $l = (\hbar/(m\omega_c))$. The density of states per unit area is $\rho_B = B/\Phi_0 = 1/(2\pi l^l)$, where $\Phi_0 = hc/e$ is the flux quantum. The wavefunction for many electrons is a Slater determinant of the Landau orbitals given above.

- Define the filling factor $\nu = \rho/\rho_B$. For integer filling factors ν, the Landau levels are full and the transverse Hall conductance is given by

$$\sigma_{xy} \equiv \frac{J_x}{E_y} = n\frac{e^2}{\hbar} \tag{3}$$

where n is the integer number of the filled Landau levels. The real quantities measured are total current $I = J_x D$ and transverse voltage $V = E_x D$, where D is the width. This gives

$$\frac{I}{V} = n\frac{e^2}{\hbar}. \tag{4}$$

Thus for filled levels in a perfect system we expect the Hall conductivity to be exact integer multiples of $\frac{e^2}{\hbar}$.

- Disorder. In the absence of a magnetic field expected to localize all states (Anderson localization). What happens in a magnetic field? Landau levels broaden. It turns out that for not too large disorder there is always at least one state that is extended even if others are localized. This one state is sufficient and the "dirt effects" of disorder are essential for the actual effect in practice!

- Gauge invariance arguments of Laughlin (1981). By considering a flux threading a closed loop, Laughlin was able to show that whenever the Fermi level is in a region between Landau levels – *where all states at the Fermi level are localized* – then the Hall conductance has *exact* integer values independent of disorder!

  See Laughlin's two-page paper and description in Phillips (also reviews).

  Because the gauge invariance arguments are so convincing, experiments on imperfect semiconductors are now the most accurate measurement of $\frac{e^2}{\hbar}$ and the international standard of resistance.

- Paradoxically, whenever one has the integer QHE, then it also follows that $\sigma_{xx} = 0$ (since the states at the Fermi level are localized) – but this means infinite conductivity! Zero electric field in the x direction is needed for the current to flow in the x direction. The resolution is that the resistivity tensor is the inverse of the conductivity tensor, and simple inversion gives $\rho_{xx} = 0$ at the same time as $\sigma_{xx} = 0$. A simple way to derive the result is (following Kittel p 540)

$$J_x = \sigma_{xx}E_x + \sigma_{xy}E_y; \quad J_y = \sigma_{yx}E_x + \sigma_{yy}E_y. \tag{5}$$

Since $J_y = 0$ one finds

$$J_x = (\sigma_{xx} + (\sigma_{xy})^2/\sigma_{yy})E_x, \tag{6}$$

and since $\sigma_{xx} = \sigma_{yy}$ it follows that the effective conductivity $\sigma_{xy})^2/\sigma_{yy}$ diverges.

4. Fractional Quantum Hall Effect

- What about magnetic fields that lead to fractional filling ν? This is a *strongly interacting system* where interactions can lead to qualitative effects! Can you see why?

- For fractions m/n where n is an odd integer, Laughlin gave the key argument (1983). One can construct a wavefunction that is fundamentally different from a Slater determinant that keeps the electrons apart better and gives lower energy than a Slater determinant for any repulsive interaction.

$$\Psi_m(z_1, z_2, \ldots, z_N) = \Pi_{jk}^N (z_j - z_k)^m exp \left[-\frac{1}{4l^2} \sum_j^N |z_j|^2 \right]. \tag{7}$$

  where $z_m = (x_m + iy_m)$ is the complex number that represents the position of electron m. Note this is antisymmetric as it should be; the final factor is the Landau gaussian and the first vanishes whenever electrons are at the same point and increases more slowly with $|(z_j - z_k)|$ than a Slater wavefunction for odd $m > 1$.

- The excitations obey fractional statistics and act like fermions with flux quanta attached.

- Jain has given arguments that these fractional states form hierarchies that are just like the integer effect.