Lecture 5

Recap: $\rho: G \to U(V)$ is an irreducible representation (irrep) if it has no nontrivial invariant subspaces ⇐ we can't block diagonalize all the $\rho(g)$ simultaneously

Schurs lemma: $\rho_1: G \to U(V_1)$ both irreducible
$\rho_2: G \to U(V_2)$

$H: V_1 \to V_2$ with $H\rho_1(g) = \rho_2(g)H$ for all $g \in G$

Then: ① either $H = 0$ or H is invertible

(2) if $\rho_1 = \rho_2 = \rho$ are the same

$[H, \rho(g)] = 0 \Rightarrow$ either $H = 0$ or $H = \lambda \, Id_V$

(2.5) if $H\rho_1(g) = \rho_2(g) H$ and $H \neq 0 \Rightarrow \rho_1 \simeq \rho_2$

are unitarily equivalent ($U\rho_1(g)U^\dagger = \rho_2(g)$ for all g)

Schur's Lemma in Quantum Mechanics

Hamiltonian H w/ some group of symmetries G

$\rho : G \to U(\mathcal{H})$ $\quad \mathcal{H}$ = Hilbert space $\{|\psi\rangle\}$

$[\rho(g), H] = 0$ since g is a symmetry

Pick a basis $\{|\psi_1\rangle, |\psi_2\rangle, \ldots\}$

$$H_{ij} = \langle \psi_i | H | \psi_j \rangle$$

$$\rho_{ij}(s) = \langle \psi_i | \rho(s) | \psi_j \rangle$$

$$\sum_j \rho_{ij}(s) H_{jk} = \sum_j H_{ij} \rho_{jk}(s)$$

$$\rho = \rho_1 \oplus \rho_2 \oplus \rho_3 \oplus \cdots$$

sum of irreducible representations

Suppose our basis block diagonalizes ρ

$$[\rho(s)] = \begin{pmatrix} [\rho_1(s)] & 0 & 0 & 0 \\ 0 & [\rho_2(s)] & 0 & 0 \\ 0 & 0 & [\rho_3(s)] & 0 \\ \vdots & & \vdots & \ddots \end{pmatrix}$$

$$[H] = \begin{pmatrix} [H_{11}] & [H_{12}] & [H_{13}] \\ [H_{21}] & \cdot & \ddots \\ \vdots & \cdot & \ddots \end{pmatrix}$$

Block-by Block $[\rho(s), H] = 0$ implies

$$[H_{ij}][\rho_j(s)] = [\rho_i(s)][H_{ij}]$$

Schurs lemma: $\rho_i \neq \rho_j \rightarrow [H_{ij}] = 0$

$$\text{if } \rho_i = \rho_j \rightarrow [H_{ij}] = \lambda \, \mathrm{Id}$$

→ states that transform in the same irreducible representation are degenerate

Ex: Nonrelativistic hydrogen atom

$$\{ |n \ell m_z\rangle, \; m_z = -\ell, \dots \ell \}$$

ℓ - labels the irrep of rotations SO(3))

Schur's lemma: $\langle n \ell m_z | H | n \ell' m_z' \rangle = E_{n\ell} \delta_{\ell\ell'} \delta_{m_z m_z'}$

Character theory:
- easy way to tell when two representations are equivalent
- tell when representations are irreducible
- count and enumerate all irreps of a group

Let G be a group $\rho : G \to U(V)$ a finite-dimensional representation

Def: the character $\chi_\rho : G \to \mathbb{C}$

$$\chi_\rho(g) = \mathrm{tr}[\rho(g)]$$

Immediately gives 3 useful properties

① if U is unitary $tr[\rho(g)] = tr[U^\dagger U \rho(g)] = tr[U\rho(g)U^\dagger]$

if $\rho_1 \cong \rho_2$ are equivalent $U\rho_1(g)U^\dagger = \rho_2(g)$ for all g

$\chi_{\rho_1} = \chi_{\rho_2} \Rightarrow$ $\boxed{\text{if } \chi_{\rho_1} \neq \chi_{\rho_2} \text{ then } \rho_1 \not\cong \rho_2}$

② $\rho \cong \rho_1 \oplus \rho_2$ $\quad [\rho(g)] = \left(\begin{array}{c|c} \rho_1(g) & 0 \\ \hline 0 & \rho_2(g) \end{array}\right)$ in some basis

$tr[\rho(g)] = tr[\rho_1(g)] + tr[\rho_2(g)]$

$$\boxed{\chi_{\rho_1\oplus\rho_2} = \chi_{\rho_1} + \chi_{\rho_2}}$$

③ Say g_1 and g_2 are conjugate $g_1 = g g_2 g^{-1}$ for some g

$\rho : G \to U(n)$

$$\chi_\rho(g_1) = tr[\rho(g_1)] = tr[\rho(g g_2 g^{-1})]$$
$$= tr[\rho(g)\rho(g_2)\rho(g^{-1})]$$
$$= tr[\rho(g_2)] = \chi_\rho(g_2)$$

Characters are constant on conjugacy classes ← "class functions"

$$C_{g_1} = \{ g' \in G \mid g' = g_i g_1 g_i^{-1} \}$$

$\chi(g_1) = \chi(g')$ for all $g' \in C_{g_1}$

Ex: The group D_2 from HW 1

$D_2 = \{E, C_{2x}, C_{2y}, C_{2z}\}$ $\quad C_{2x}^2 = C_{2y}^2 = C_{2z}^2 = E$

$C_{2x}C_{2y} = C_{2z} = C_{2y}C_{2x}$

vector representation ρ_V

$\rho_V(E) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad \rho_V(C_{2y}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$\rho_V(C_{2x}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ $\quad \rho_V(C_{2z}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Abelian group $\quad g g' g^{-1} = g' \Rightarrow$ every element in its own conjugacy class

	E	C_{2x}	C_{2y}	C_{2z}
χ_{ρ_V}	3	-1	-1	-1

invariant subspaces:

$$\left\{\begin{pmatrix} b_1 \\ 0 \\ 0 \end{pmatrix}\right\}, \left\{\begin{pmatrix} 0 \\ b_2 \\ 0 \end{pmatrix}\right\}, \left\{\begin{pmatrix} 0 \\ 0 \\ b_3 \end{pmatrix}\right\}$$

$$\rho_V = \rho_{B_1} \oplus \rho_{B_2} \oplus \rho_{B_3}$$

conjugacy classes

	E	C_{2x}	C_{2y}	C_{2z}
χ_{B_1}	1	1	-1	-1
χ_{B_2}	1	-1	1	-1
χ_{B_3}	1	-1	-1	1

irreps

"Character table"

Schur's Lemma ⇒ Schur Orthogonality relations

Let G be a group, $\rho_1 : G \to U(V_1)$, $\rho_2 : G \to U(V_2)$ irreducible, finite-dimensional

and A be <u>any</u> $\dim V_2 \times \dim V_1$ matrix

$A : V_1 \to V_2$

Trick: average over group elements

$A_G = \sum_{g \in G} \rho_2(g^{-1}) A \rho_1(g)$ Claim: $A_G \rho_1(g) = \rho_2(g) A_G$

$$A_G \rho_1(g') = \sum_{g \in G} \rho_2(g^{-1}) A \rho_1(g) \rho_1(g')$$

$$= \sum_{g \in G} \rho_2(g^{-1}) A \rho_1(gg')$$

$g'' = gg'$

$g^{-1} = (g'' g'^{-1})^{-1}$

$= g' g''^{-1}$

$$= \sum_{g'' \in G} \rho_2(g' g''^{-1}) A \rho_1(g'')$$

$$= \rho_2(g') A_G$$

Schur Lemma:

$$A_G = \begin{cases} 0 & \text{if } \rho_1 \neq \rho_2 \\ \lambda \, \mathrm{Id} & \text{if } \rho_1 = \rho_2 \end{cases}$$

Lets apply this to a special

jth column

$$A = E_{ij} = \begin{pmatrix} 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ & & & & & \vdots \\ & & 0 & \cdots & & \end{pmatrix}$$ ith row

index rows and columns

$$[E_{ij}]^{\mu\nu} = \delta_i^\mu \delta_j^\nu$$

Average over group elements

$$[E_{ij}]_G^{\mu\nu} = \sum_{g\in G}\sum_{\alpha\beta}[\rho_2(g^{-1})]^{\mu\alpha}[E_{ij}]^{\alpha\beta}[\rho_1(g)]^{\beta\nu}$$

$$= \sum_{g\in G}[\rho_2(g^{-1})]^{\mu i}[\rho_1(g)]^{j\nu} = \begin{cases} 0 & \text{if } \rho_1 \not\cong \rho_2 \\ \lambda\delta^{\mu\nu} & \text{if } \rho_1 = \rho_2 \end{cases}$$

To find λ let $\rho_1 = \rho_2$ and take a trace

$$\sum_{\mu} \lambda \delta_{\mu\mu} = \lambda \dim \rho_1 \quad \leftarrow \text{RHS}$$

$$\text{LHS: } \sum_{g \in G} \sum_{\mu} [\rho_1(g^{-1})]^{\mu i} [\rho_1(g)]^{j\mu} = \sum_{g \in G} \delta_{ij} = |G| \delta_{ij}$$

$$\lambda = \frac{|G|}{\dim \rho_1} \delta_{ij}$$

Schur Orthogonality Relation: let ρ_1 and ρ_2 be irreps of a finite group then

$$\sum_{g \in G} [\rho_2(g^{-1})]^{\mu i} [\rho_1(g)]^{j\nu} = \begin{cases} 0 & \text{if } \rho_1 \not\cong \rho_2 \\ \frac{|G|}{\dim \rho_1} \delta_{ij} \delta^{\mu\nu} & \text{if } \rho_1 = \rho_2 \end{cases}$$

To get a statement about characters, take traces

$$\sum_{g\in G}\sum_{\mu\nu}[\rho_2(g^{-1})]^{\mu\mu}[\rho_1(g)]^{\nu\nu} = \begin{cases} 0 & \text{if } \rho_1 \neq \rho_2 \\ |G| & \text{if } \rho_1 = \rho_2 \end{cases}$$

$$\parallel$$

$$\sum_{g\in G}\chi_{\rho_2}(g^{-1})\chi_{\rho_1}(g)$$

$$\rho_2(g^{-1}) = [\rho_2(g)]^\dagger$$

$$\chi_{\rho_2}(g^{-1}) = (\chi_{\rho_2}(g))^*$$

$$\Rightarrow \underline{\frac{1}{|G|}}\sum_{g\in G}(\chi_{\rho_2}(g))^*(\chi_{\rho_1}(g)) = \begin{cases} 0 & \text{if } \rho_1 \neq \rho_2 \\ 1 & \text{if } \rho_1 = \rho_2 \end{cases}$$

$$|||$$

$$\langle \chi_{\rho_2}, \chi_{\rho_1} \rangle$$

Characters for irreps are orthonormal under this dot product!

Say we have a reducible representation

$$\rho \cong \underbrace{\rho_1 \oplus \rho_1 \oplus \dots \oplus \rho_1}_{n_1} \oplus \underbrace{\rho_2 \oplus \rho_2 \oplus \dots}_{n_2} \dots$$

$$\sum_{g \in G} \chi_{\rho_1}(g^{-1}) \rho(g)$$

Next time

$$\chi_\rho = n_1 \chi_{\rho_1} + n_2 \chi_{\rho_2} + \dots = \sum_{i=1}^{N_{tot}} n_i \chi_{\rho_i}$$ ← irreducible

$$\langle \chi_{\rho_j}, \chi_\rho \rangle = n_j$$ ← multiplicities

irreducible reducible

Check: D_2

		E	C_{2x}	C_{2y}	C_{2z}
irreps	χ_{B_1}	1	1	-1	-1
	χ_{B_2}	1	-1	1	-1
	χ_{B_3}	1	-1	-1	1
	χ_A	1	1	1	1

Note: Every group has one irrep we can always write:

$\rho_A(g) = 1$ $\rho_A(g_1 g_2) = 1 = \rho_A(g_1)\rho_A(g_2)$