Chapter 5

Quantum Electrodynamics

1. EM Interactions of spin-0 Particles

We now consider electromagnetic interactions involving spin-0 particles. These particles are assumed to be structureless point-like charged particles. We would like to describe scattering processes such as

$$\begin{aligned} &\pi^+\pi^- \to \pi^+\pi^- \\ &\pi^+\pi^- \to K^+K^- \\ &\pi^+ A \to \pi^+ A \end{aligned} \tag{5.1}$$

or a fictitious 'spinless' electron scattering off a spinless electron or muon

$$\begin{aligned} &'e^-{}''e^-{}' \to 'e^-{}'+'e^-{}' \\ &e^-\mu^- \to e^-\mu^- \\ &e^-e^+ \to \mu^-\mu^+ \qquad \text{etc.} \end{aligned} \tag{5.2}$$

To study the transition rate and cross section of these processes, we start by considering the lowest order perturbation theory. The transition amplitude T_{fi} is given as

$$T_{fi} = -i\int d^4x\ \phi_f^*(x)\ v(x)\ \phi_i(x) \tag{5.3}$$

For spin-0 particles, the Klein-Gordon equation

$$\left(\partial_\mu\partial^\mu + m^2\right)\ \phi(x) = 0 \tag{5.4}$$

gives the following plane-wave solutions

$$\begin{aligned} \phi_i(x) &= N_i e^{-iP_i\bullet x} \\ \phi_f^*(x) &= N_f e^{iP_f\bullet x} \end{aligned} \tag{5.5}$$

Interaction of a charged particle in an EM potential $A^\mu = \left(A^0, \vec{A}\right)$ is obtained by the substitution

$$i\partial^\mu \rightarrow i\partial^\mu + eA^\mu \tag{5.6}$$

The Klein-Gordon equation becomes

$$\left(\partial_\mu \partial^\mu + V + m^2\right) \phi(x) = 0 \tag{5.7}$$

where $$V = -ie\left(\partial_\mu A^\mu + A^\mu \partial_\mu\right) - e^2 A^2$$

Substituting Equation 5.7 into Equation 5.3 and ignoring the higher order (in e) term e^2A^2, we obtain

$$T_{fi} = -e\int \phi_f^* \left(A^\mu \partial_\mu + \partial_\mu A^\mu\right) \phi_i \; d^4x \tag{5.8}$$

Integration by part changes the second term of Equation 5.8 into

$$\int \phi_f^* \partial_\mu \left(A^\mu \phi_i\right) \; d^4x = -\int \partial_\mu \left(\phi_f^*\right) A^\mu \phi_i d^4x \tag{5.9}$$

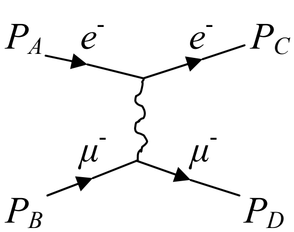
and T_{fi} can be written as

$$T_{fi} = -i\int j_\mu^{fi} A^\mu d^4x \tag{5.10}$$

where the transition current j_μ^{fi} between states i and f is

$$j_\mu^{fi} = -ie\left[\phi_f^* \left(\partial_\mu \phi_i\right) - \left(\partial_\mu \phi_f^*\right)\phi_i\right] \tag{5.11}$$

Consider a 2 → 2 process such as spin-less $e^-\mu^- \rightarrow e^-\mu^-$ scattering



$$j_\mu^{fi} = -eN_A N_C \left(P_A + P_C\right)_\mu e^{i(P_C - P_A)\cdot x}$$
or
$$j_\mu^{(1)} = -eN_A N_C \left(P_A + P_C\right)_\mu e^{i(P_C - P_A)\cdot x} \tag{5.12}$$

What is the EM potential A^μ generated by μ^-? The Maxwell Equation under the Lorentz condition, $\partial_\mu A^\mu = 0$, becomes

$$\square^2 A^\mu = j^\mu \tag{5.13}$$

The current j^μ of the muon is analogous to that of the electron, and is given as

$$j^\mu_{(2)} = -eN_B N_D \left(P_B + P_D\right)^\mu e^{i(P_D - P_B)x} \tag{5.14}$$

Keeping in mind that

$$\square^2 e^{+iq\bullet x} = -q^2 e^{+iq\bullet x} \tag{5.15}$$

The solution to Equation 5.13 can be obtained by inspection:

$$A^\mu = -\frac{1}{q^2} j^\mu_{(2)} \tag{5.16}$$

where

$$q = P_D - P_B$$

The Transition amplitude becomes

$$T_{fi} = -i\int j^{(1)}_\mu (x)\left(-\frac{1}{q^2}\right) j^{(\mu)}_2 (x) d^4x \tag{5.17}$$

Substituting Equations 5.12 and 5.14 into Equation 5.17 and noting that

$$\int e^{i(P_C - P_A + P_D - P_B)\bullet x} d^4x = (2\pi)^4 \delta^{(4)}\left(P_D + P_C - P_B - P_A\right) \tag{5.18}$$

we obtain

$$T_{fi} = -iN_A N_B N_C N_D = (2\pi)^4 \delta^{(4)}\left(P_D + P_C - P_B - P_A\right) M \tag{5.19}$$

where

$$-iM = \left[ie\left(P_A + P_C\right)^\mu\right]\left(-i\frac{g_{\mu\nu}}{q^2}\right)\left[ie\left(P_B + P_D\right)^\nu\right] \tag{5.20}$$

M is Lorentz invariant and called the ‘invariant amplitude’.

For $A + B \rightarrow C + D$, the transitions per unit time per unit volume is

$$W_{fi} = \frac{\left|T_{fi}\right|^2}{T \bullet V} \tag{5.21}$$

It can be shown that the transition time *T* and the volume *V* in Equation 5.21 cancel the delta function in T_{fi} specifically.

$$T_{fi} \;\alpha \left(2\pi\right)^4 \delta^4\left(P_D + P_C - P_B - P_A\right) \tag{5.22}$$

and

$$\left|T_{fi}\right|^2 \alpha \; \left(2\pi\right)^8 \delta^4\left(P_D + P_C - P_B - P_A\right)\; \delta^4\left(P_D + P_C - P_B - P_A\right) \tag{5.23}$$

Consider the 0-th component of the delta function in Equation 5.23:

$$\begin{aligned} &\left(2\pi\right)^2 \delta\left(\mathrm{E}_F - \mathrm{E}_I\right)\delta\left(\mathrm{E}_F - \mathrm{E}_I\right) \\ &= \left(2\pi\right)\; \delta\left(\mathrm{E}_F - \mathrm{E}_I\right)\int_{-T/2}^{T/2} e^{i\left(\mathrm{E}_F - \mathrm{E}_I\right)t}\,dt \\ &= \left(2\pi\right)\; \delta\left(\mathrm{E}_F - \mathrm{E}_I\right) \bullet 2 \bullet \frac{\sin\left[\left(T/2\right)\left(\mathrm{E}_F - \mathrm{E}_I\right)\right]}{\mathrm{E}_F - \mathrm{E}_I} \end{aligned} \tag{5.24}$$

The first delta function in Equation 5.24 requires

$$\frac{\sin\left[\left(T/2\right)\left(\mathrm{E}_F - \mathrm{E}_I\right)\right]}{\mathrm{E}_F - \mathrm{E}_I} = \frac{T}{2} \tag{5.25}$$

and Equation 5.24 becomes

$$\left(2\pi\right)^2 \delta\left(\mathrm{E}_F - \mathrm{E}_I\right)\; \delta\left(\mathrm{E}_F - \mathrm{E}_I\right) = \left(2\pi\right)\; \delta\left(\mathrm{E}_F - \mathrm{E}_I\right)T \tag{5.26}$$

Similarly, one can show that

$$\begin{aligned} &\left|T_{fi}\right|^2 \alpha \left(2\pi\right)^4 \delta^4\left(P_D + P_C - P_B - P_A\right)\left(T\right)\left(L_x\right)\left(L_y\right)\left(L_z\right) \\ &= \left(2\pi\right)^4 \delta^4\left(P_D + P_C - P_B - P_A\right) \bullet T \bullet V \end{aligned} \tag{5.27}$$

To convert W_{fi} into the cross section, $d\sigma$, which characterized the effective area within which the $A + B$ collision can lead to $C + D$, one needs to multiply W_{fi} by the number of final states and divide it by the initial flux:

$$d\sigma = \frac{W_{fi}}{(\text{initial flux})} \times (\text{number of final states}) \tag{5.28}$$

Adopting the so-called covariant normalization for the Klein-Gordon equation

$$\int_V \zeta\, dv = 2\text{E} \qquad\qquad N = \frac{1}{\sqrt{v}} \tag{5.29}$$

The initial flux is therefore proportional to the number of beam particles passing through unit area per unit time, $|V_A| 2\text{E}_A / V$, and the number of target particles per unit volume, $2\text{E}_B/\text{V}$.

$$\text{Initial Flux} = |\vec{V}_A| \frac{2E_A}{V}\frac{2E_B}{V} \tag{5.30}$$

The number of final states in a volume V with momentum within $\alpha^3 P$ is $V\,\alpha^3 P/(2\pi)^3$. Since there are 2E particles in V, we have

$$\text{Number of final states / particle} = \frac{Vd^3P}{(2\pi)^3\, 2\text{E}} \tag{5.31}$$

And the number of available final states for particles C, D scattered into $\alpha^3 P_C$, $\alpha^3 P_D$ is

$$\frac{Vd^3P_C}{(2\pi)^3\, 2\text{E}_C}\;\frac{Vd^3P_D}{(2\pi)^3\, 2\text{E}_D} \tag{5.32}$$

Inserting Equations 5.21, 5.27, 5.19, 5.30, 5.32 into Equation 5.28, we finally obtain

$$d\sigma = \frac{|M|^2}{F} dQ \tag{5.33}$$

where

$$dQ = (2\pi)^4 \delta^{(4)}(P_C + P_D - P_A - P_B) \frac{d^3P_C}{(2\pi)^3 2E_C} \frac{d^3P_D}{(2\pi)^3 2E_D} \tag{5.34}$$

is the Lorentz invariant phase space factor (αLips) and the flux factor F is

$$F = |V_A| \bullet 2E_A \bullet 2E_B \tag{5.35}$$

in the lab frame.

For a general collinear collision between A and B

$$\begin{aligned} F &= \left|\vec{V}_A - \vec{V}_B\right| \bullet 2E_A \bullet 2E_B = \left|\vec{V}_A\right| + \left|\vec{V}_B\right| \bullet 2E_A \bullet 2E_B \\ &= 4\left(\left|\vec{P}_A\right| E_B + \left|\vec{P}_B\right| E_A\right) \qquad \left(\left|\vec{V}\right| = \frac{\left|\vec{P}\right|}{E}\right) \\ &= 4\left[(P_A \bullet P_B)^2 - M_A^2 M_B^2\right]^{1/2} \end{aligned} \tag{5.36}$$

In the center-of-mass frame for the process $A + B \rightarrow C + D$, one can show (Ex. 4.2 of H & M)

$$dQ = \frac{1}{4\pi^2} \frac{P_f}{4\sqrt{S}} d\Omega \tag{5.37}$$

S is the square of center-of-mass energy

and from 5.36,

$$F = 4P_i\sqrt{S} \tag{5.38}$$

where P_i, P_f are the initial and final 3-momentum in the C.M. frame.

Equations 5.33, 5.37, 5.38 give the following important expression for the differential cross-section in the C.M. frame:

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{1}{64\pi^2 S} \frac{P_f}{P_i} |M|^2 \tag{5.39}$$

<u>Note added:</u>

Here is the derivation for Equation 5.37:

$$dQ^{(6)} = (2\pi)^4 \delta^4 (P_3 + P_4 - P_1 - P_2) \; \frac{d^3P_3 \; d^3P_4}{(2\pi)^3 2E_3 (2\pi)^3 2E_4}$$

To evaluate dQ in the C.M. frame $\left(\vec{P}_1 = -\vec{P}_2,\ \vec{P}_3 = -\vec{P}_4\right)$, we first integrate over d^3P_4:

$$dQ^{(3)} = \frac{\delta(E_3 + E_4 - W)}{16\pi^2 E_3 E_4} \; d^3P_3$$

where $$W = E_1 + E_2 = E_3 + E_4 = E_{C.M.} = \sqrt{S}$$

To proceed further, we need to express d^3P_3 in terms of dE_3 and express E_4 in terms of E_3:

$$d^3P_3 = P_3^2 dP_3 d\Omega = P_3 E_3 dE_3 d\Omega$$

since $$P_3 dP_3 = E_3 dE_3 \left(E_3^2 = P_3^2 + M_3^2\right)$$

$$E_3 + E_4 - W = E_3 + \left(E_3^2 - M_3^2 + M_4^2\right)^{1/2} - W$$

since $$|P_3| = |P_4|,\ \ E_3^2 - M_3^2 = E_4^2 - M_4^2$$

Therefore $$dQ = \int \frac{\delta\left[E_3 + \left(E_3^2 - M_3^2 + M_4^2\right)^{1/2} - W\right] P_3 dE_3 d\Omega}{16\pi^2 E_4}$$

Now, $$\int dE_3 \delta\left(g(E_3)\right] = \left|\frac{dg}{dE_3}\right|^{-1}$$

$$g(E_3) = E_3 + \left(E_3^2 - M_3^2 + M_4^2\right)^{1/2} - W$$

$$\frac{dg}{d\mathrm{E}_3} = 1 + \frac{\mathrm{E}_3}{\mathrm{E}_4} = \frac{W}{\mathrm{E}_4}$$

Here $$\left(dQ\right)_{C.M.} = \frac{P_3 d\Omega}{16\pi^2 W} = \frac{P_f d\Omega}{16\pi^2 \sqrt{S}}$$

Now, consider dQ in the lab frame:

$$P_2 = (M_2, 0)$$

Integrating over d^3P_4, we have

$$dQ = \int \frac{\delta\left(\mathrm{E}_3 + \mathrm{E}_4 - \mathrm{E}_1 - M_2\right) P_3 d\ \mathrm{E}_3 d\Omega}{16\pi^2 \mathrm{E}_4}$$

Now, $$\vec{P}_4 = \vec{P}_1 - \vec{P}_3 \qquad \left(\text{since} \quad \vec{P}_2 = 0\right)$$

$$\left|\vec{P}_4\right|^2 = \left|\vec{P}_1 - \vec{P}_3\right|$$

$$\mathrm{E}_4^2 = M_4^2 + P_1^2 + P_3^2 - 2P_1P_3\cos\theta$$

For the δ-function, we have

$$g\left(\mathrm{E}_3\right) = \mathrm{E}_3 + \left(M_4^2 + P_1^2 + P_3^2 - 2P_1P_3\cos\theta\right)^{1/2} - \mathrm{E}_1 - M_2$$

$$\frac{dg}{d\mathrm{E}_3} = 1 + \left[2\mathrm{E}_3 - 2\left(\frac{P_1\mathrm{E}_3}{P_3}\right)\cos\theta\right] \Big/ 2\mathrm{E}_4$$

$$= \left[\mathrm{E}_1 + M_2 - \left(\frac{P_1\mathrm{E}_3}{P_3}\right)\cos\theta\right] \Big/ \mathrm{E}_4$$

Hence, $$\left(dQ\right)_{\text{lab}} = \frac{P_3 d\Omega}{16\pi^2\left[\mathrm{E}_1 + M_2 - \left(\frac{P_1\mathrm{E}_3}{P_3}\right)\cos\theta\right]}$$

1.1 Mandelstam Variables

Before examining the $A + B \rightarrow C + D$ process in some detail, it is useful to consider the variables specifying such a reaction. There are various choices for variables, such as beam energy and scattering angle. However, it is advantageous to specify variables which are Lorentz invariant quantities. The Mandelstam variables (s, t and u) are defined as

$$\begin{aligned} s &= (P_A + P_B)^2 = (P_C + P_D)^2 && \text{total C.M. energy squared} \\ t &= (P_A - P_C)^2 = (P_D - P_B)^2 && \text{four momentum transfer squared} \\ u &= (P_A - P_D)^2 = (P_C - P_B)^2 && \end{aligned} \quad (5.40)$$

Note that $P_A + P_B = P_C + P_D$.

s, t, u are not independent variables, since

$$s + t + u = M_A^2 + M_B^2 + M_C^2 + M_D^2 \quad (5.41)$$

If $M_A = M_B = M_C = M_D = M$ ($e^-e^- \rightarrow e^-e^-$, $\pi^+\pi^+ \rightarrow \pi^+\pi^+$ for example), then in the C.M. frame we have

$$\begin{aligned} s &= 4(P^2 + M^2) \\ t &= -2P^2(1 - \cos\theta) \\ u &= -2P^2(1 + \cos\theta) \end{aligned} \quad (5.42)$$

where P is the 3-momentum in the C.M. frame, and θ is the C.M. scattering angle. Note that $s > 0$, $t \leq 0$, $u \leq 0$.

The Mandelstam variables are very convenient in expressing one scattering process in terms of another related scattering process.

If one expresses the amplitude M for the process

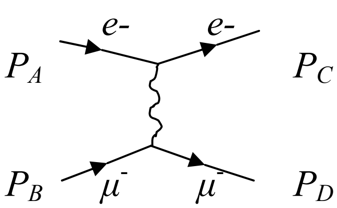
a) $P_A + P_B \rightarrow P_C + P_D$
as $M(s, t, u)$, then the other related processes have the amplitudes as follows:
b) $P_A + P_B \rightarrow P_D + P_C$: $M(s, u, t)$
c) $P_A + (-P_C) \rightarrow (-P_B) + P_D$: $M(t, s, u)$
d) $P_A + (-P_D) \rightarrow P_C + (-P_B)$: $M(u, t, s)$
e) $(-P_C) + (-P_D) \rightarrow (-P_A) + (-P_B)$: $M(s, t, u)$

As an example, take reaction a) as $e^-\mu^- \rightarrow e^-\mu^-$, then

a) $e^-\mu^- \rightarrow e^-\mu^-$: $M(s, t, u)$
b) $e^-\mu^- \rightarrow \mu^-e^-$: $M(s, u, t)$
c) $e^-e^+ \rightarrow \mu^+\mu^-$: $M(t, s, u)$
d) $e^-\mu^+ \rightarrow e^-\mu^+$: $M(u, t, s)$
e) $e^+\mu^+ \rightarrow e^+\mu^+$: $M(s, t, u)$

1.2 Spinless $e^-\mu^- \rightarrow e^-\mu^-$ Scattering

Now we consider the invariant amplitude M and the scattering cross-section for a 'spinless' $e^-\mu^- \rightarrow e^-\mu^-$ process.



Recall Equation 5.20

$$-iM = \left[ie\left(P_A + P_C\right)^{\mu}\right]\left(-i\frac{g_{\mu\nu}}{q^2}\right)\left[ie\left(P_B + P_D\right)^{\nu}\right]$$

first $e\gamma$ vertex; photon propagator; second $e\gamma$ vertex

$$M = -e\,\left(P_A + P_C\right) \bullet \left(P_B + P_D\right) \bullet \frac{1}{t} \qquad (5.43)$$

Now $\quad P_A + P_C = (P_A + P_B) + (P_C - P_B) = (P_A + P_B) + (P_A - P_D)$
$\quad P_B + P_D = (P_A + P_B) - (P_A - P_D)$

Therefore
$$M = -e^2\left[\left(P_A+P_B\right)^2-\left(P_A-P_D\right)^2\right]\bullet\frac{1}{t} \qquad (5.44)$$
$$= -e^2\left(s-u\right)/t$$

Equations 5.39 and 5.44 give

$$\left.\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{1}{64\pi^2 s}\;\frac{P_f}{P_i}\left(e^4\frac{\left(s-u\right)^2}{t^2}\right) \qquad (5.45)$$
$$= \frac{\alpha^2}{4s}\;\frac{P_f}{P_i}\;\frac{\left(s-u\right)^2}{t^2}$$

At high energies, masses are neglected, $P_i = P_f = P$ and

$$s = 4\left(P^2+M^2\right) \simeq 4P^2$$
$$t = -2P^2\left(1-\cos\theta\right)$$
$$u = -2P^2\left(1+\cos\theta\right)$$

Hence,

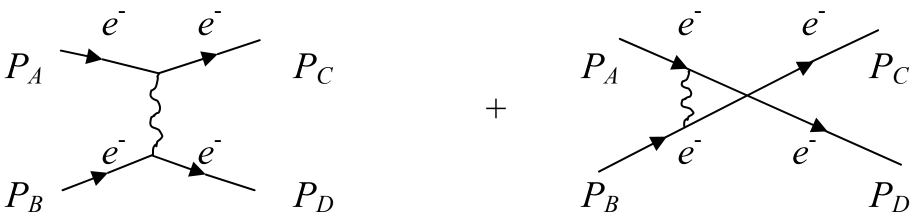
$$\frac{\left(s-u\right)^2}{t^2} = \frac{\left[4P^2+2P^2\left(1+\cos\theta\right)\right]^2}{\left[-2P^2\left(1-\cos\theta\right)\right]^2} = \frac{\left(3+\cos\theta\right)^2}{\left(1-\cos\theta\right)^2} \qquad (5.46)$$
$$= \frac{\left(2+2\cos^2\theta/2\right)^2}{\left(2\sin^2\theta/2\right)^2} = \frac{\left(1+\cos^2\theta/2\right)^2}{\sin^4\theta/2}$$

and

$$\left.\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{\alpha^2}{4s}\left(\frac{1+\cos^2\theta/2}{\sin^2\theta/2}\right)^2$$

1.3 Spinless $e^-e^- \to e^-e^-$

Now consider spinless $e^-e^- \to e^-e^-$ scattering. There are two diagrams contributing to this process:



The first diagram is analogous to the diagram we considered earlier for $e^-\mu^- \rightarrow e^-\mu^-$. The second diagram reflects the fact that one can not tell if P_C originates from P_A or from P_B.

The invariant amplitude is a sum of these two diagrams

$$-iM_{e^-e^-} = -i\left[-e^2\frac{(P_A+P_C)\bullet(P_B+P_D)}{(P_D-P_B)^2} - e^2\frac{(P_A+P_D)\bullet(P_B+P_C)}{(P_C-P_B)^2}\right] \quad (5.48)$$

Note that $M_{e\text{-}e\text{-}}$ is now symmetric with respect to the exchange of $P_C \leftrightarrow P_D$, as well as the exchange of $P_A \leftrightarrow P_B$. This is a consequence that we assume e^- is spinless and following Bose statistics. Otherwise, the amplitude should be antisymmetric with respect to these exchanges.

In terms of the Mandelstam variables, Equation 5.48 can be expressed as

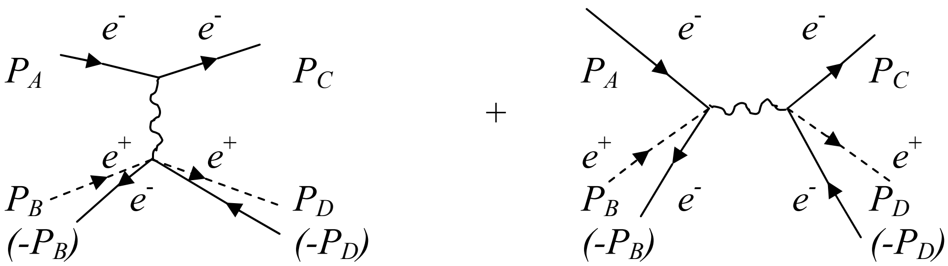
$$M_{e^-e^-} = -e^2\left(\frac{s-u}{t}\right) - e^2\left(\frac{s-t}{u}\right) \quad (5.49)$$

The second term in Equation 5.49 is obtained by $t \leftrightarrow u$ exchange in the first term, as one expects.

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{\alpha^2}{s}\,\frac{(4-\sin^2\theta)^2}{\sin^4\theta}$$

1.4 Spinless $e^-e^+ \rightarrow e^-e^+$

There are two diagrams contributing to this reaction:



The first diagram is the exchange diagram analogous to $e^-\mu^- \to e^-\mu^-$. The second diagram is an annihilation diagram. Note that the e^+ lines are replaced by e^- lines with opposite momenta. The corresponding invariant amplitudes are

$$-iM_{e^-e^+} = -i\left[-e^2\frac{(P_A+P_C)\bullet(-P_D-P_B)}{(P_D-P_B)^2} - e^2\frac{(P_A-P_B)\bullet(-P_D+P_C)}{(P_C-P_D)^2}\right] \tag{5.50}$$

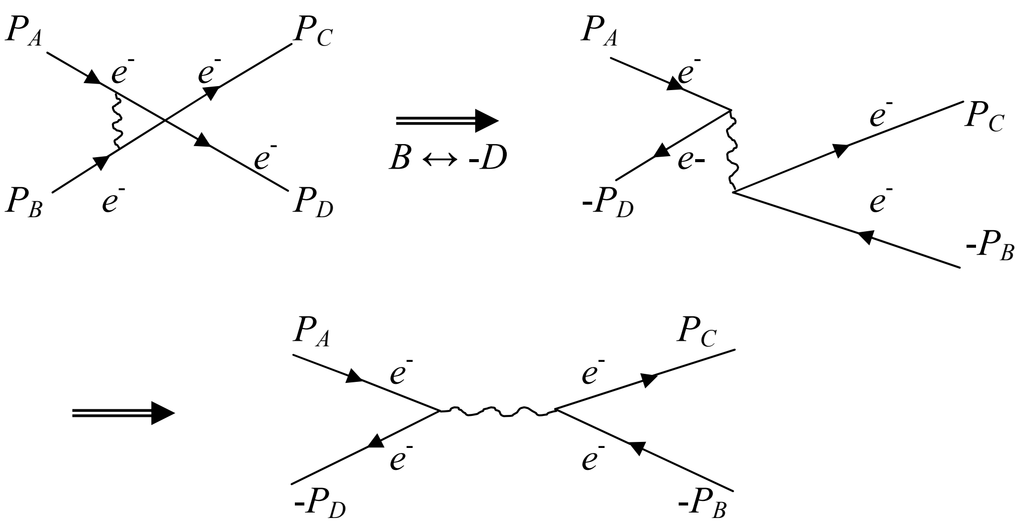
At each vertex $P + P'$ corresponds to the incoming e^- momentum P and the outgoing e^- momentum P'.

In terms of the Mandelstam variables, Equation 5.50 can be written as

$$\begin{aligned} M_{e^-e^+} &= -e^2\left(\frac{u-s}{t}\right) - e^2\left(\frac{u-t}{s}\right) \\ &= e^2\left(\frac{s-u}{t}\right) + e^2\left(\frac{t-u}{s}\right) \end{aligned} \tag{5.51}$$

Equation 5.51 can be obtained from Equation 5.49 ($e^-e^- \to e^-e^-$) by interchanging ($s \leftrightarrow u$).

Although the annihilation diagram for $e^-e^+ \to e^-e^+$ has a different appearance compared with the second exchange diagram in the $e^-e^- \to e^-e^-$ reaction, these two diagrams are actually related by the $B \leftrightarrow -D$ interchange. This can be seen graphically by interchanging $B \leftrightarrow - D$ for the $e^-e^- \to e^-e^-$ diagram.



which ends up as the annihilation diagram for the $e^-e^+ \rightarrow e^-e^+$.

It is interesting to note that the cross-section for $e^-e^- \rightarrow e^-e^-$ scattering (Equation 5.49) diverges at $\theta = 0^o$ and $\theta = 180^o$, corresponding to $t = 0$ and $u = 0$. Since t and u are the invariant masses of the exchanged virtual photons for the two diagrams of the $e^-e^- \rightarrow e^-e^-$ scattering, a vanishing mass of the virtual photon implies that the range of the interaction becomes infinite. Hence the cross-section diverges.

For the annihilation diagram, the corresponding amplitude does not diverge, since the virtual photon has an invariant mass greater than $2M_e$ and cannot be zero.

2. EM Interactions of spin-½ particles

We follow a similar procedure as the spin-0 case to obtain the expression for the invariant amplitude.

For a spin-½ charged particle interacting with an EM field, the Dirac equation

$$\left(\gamma_\mu P^\mu - m\right)\psi = 0 \tag{5.52}$$

becomes (after the $P^\mu \rightarrow P^\mu + eA^\mu$ substitution)

$$\left(\gamma_\mu P^\mu - m\right)\psi = \rightarrow \left(-e\gamma_\mu A^\mu\right)\psi = \left(\gamma^0 V\right)\psi \tag{5.53}$$

where

$$\gamma^0 V = -e\gamma_\mu A^\mu \tag{5.54}$$

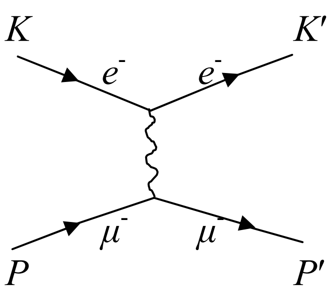
The transition amplitude T_{fi} is given as

$$\begin{aligned} T_{fi} &= -i\int \psi_f^+(x) V(x) \psi_i(x) d^4x \\ &= ie\int \overline{\psi}_f(x)\gamma_\mu A^\mu \psi_i(x) d^4x \\ &= -i\int j_\mu A^\mu d^4x \end{aligned} \tag{5.55}$$

where the current density j_μ for the $i \to f$ transition is

$$j_\mu = -e\overline{\psi}_f \gamma_\mu \psi_i \tag{5.56}$$

Now, consider the $e^- \mu^- \to e^- \mu^-$ scattering (with spin-½ e^- and μ^-)



Following similar steps as for the Klein-Gordon equation, one can deduce

$$-iM = \left(ie\overline{u}(K')\gamma^\mu u(K)\right)\left(\frac{ig_{\mu\nu}}{q^2}\right)\left(ie\overline{u}(P')\gamma_\nu u(P)\right)$$

first $e\gamma$ vertex; γ-propagator; second $\mu\gamma$ vertex

2.1 $e^-\mu^- \to e^-\mu^-$ Scattering

$$M = -e^2\overline{u}(K')\gamma^\mu u(K)\frac{1}{q^2}\overline{u}(P')\gamma_\mu u(P) \tag{5.58}$$

For measurements using unpolarized e^- and μ^-, the scattering cross-section should be an incoherent sum over the various spin states of e^-, μ^-, and averaged over the initial e^-, μ^- spins:

$$\overline{|M|^2} = \frac{1}{(2S_A+1)(2S_B+1)}\sum_{\substack{\text{spin}\\\text{states}}}|M|^2 = \frac{1}{4}\sum_{\substack{\text{spin}\\\text{states}}}|M|^2 \tag{5.59}$$

Equations 5.58 and 5.59 give

$$\overline{|M|^2} = \frac{1}{4}\sum_{spin}\frac{e^4}{q^4}\left(\bar{u}(K')\gamma^{\mu}u(K)\bar{u}(P')\gamma_{\mu}u(P)\right)\left(\bar{u}(K')\gamma^{\nu}u(K)\bar{u}(P')\gamma_{\nu}u(P)\right)^* \quad (5.60)$$

$\overline{|M|^2}$ can be viewed as a contraction of two lepton tensors

$$\overline{|M|^2} = \frac{e^4}{q^4}L_e^{\mu\nu}L_{\mu\nu}^{muon} \quad (5.61)$$

For the electron tensor, $L_e^{\mu\nu}$, we have

$$L_e^{\mu\nu} = \frac{1}{2}\sum_{s,s'}\left(\bar{u}^{s'}(K')\gamma^{\mu}u^{s}(K)\right)\left(\bar{u}^{s'}(K')\gamma^{\nu}u^{s}(K)\right)^* \quad (5.62)$$

Since $\bar{u}(K')\gamma^{\nu}u(K)$ is a number, its complex conjugate is identical to its Hermitian conjugate. Therefore

$$\left[\bar{u}^{s'}(K')\gamma^{\nu}u^{s}(K)\right]^{+} = u^{s}(K)^{+}\left(\gamma^{\nu}\right)^{+}\gamma^{0}u^{s'}(K') = \bar{u}^{s}(K)\gamma^{\nu}u^{s'}(K') \quad (5.63)$$

where we have used $\left(\gamma^{\nu}\right)^{+}\gamma^{0} = \gamma^{0}\gamma^{\nu}$ relation.

Equation 5.63 shows that the operation of complex conjugate on $\bar{u}(K')\gamma^{\nu}u(K)$ is simply equivalent to interchanging (K, S) and (K′, S′).

From Equations 5.62 and 5.63, we obtain

$$L_e^{\mu\nu} = \frac{1}{2}\sum_{s,s'}\left(\bar{u}^{s'}(K')\gamma^{\mu}u^{s}(K)\right)\left(\bar{u}^{s}(K)\gamma^{\nu}u^{s'}(K')\right) \quad (5.64)$$

Each term on the right-hand side of Equation 5.64 is a product of two numbers. It is useful to view Equation 5.64 in a somewhat different fashion:

$$\bar{u}^{s'}(K')\gamma^{\mu}u^{s}(K)\bar{u}^{s}(K)\gamma^{\nu}u^{s'}(K') = \left[\bar{u}^{s'}(K')\gamma^{\mu}u^{s}(K)\bar{u}^{s}(K)\gamma^{\nu}\right]\left[u^{s'}(K')\right] \quad (5.65)$$

Now, the right-hand side of Equation 5.65 corresponds to a product of a column 4 x 1 matrix by a row 1 x 4 matrix:

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = A\bullet B \quad (5.66)$$

One can invert the order of A and B, and Equation 5.66 becomes

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \bullet \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \end{pmatrix} = \begin{pmatrix} a_1b_1 & a_2b_1 & a_3b_1 & a_4b_1 \\ a_1b_2 & a_2b_2 & a_3b_2 & a_4b_2 \\ a_1b_3 & a_2b_3 & a_3b_3 & a_4b_3 \\ a_1b_4 & a_2b_4 & a_3b_4 & a_4b_4 \end{pmatrix} \quad (5.67)$$

Therefore

$$A \bullet B = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \end{pmatrix} \bullet \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = T_r \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \bullet \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \end{pmatrix} \quad (5.68)$$

Using Equation 5.68, Equation 5.65 becomes (after moving $u^{s'}(K')$ to the front)

$$\bar{u}^{s'}(K')\gamma^{\mu}u^{(s)}(K)\bar{u}^{s}(K)\gamma^{\nu}u^{s'}(K') = \left[u^{s'}(K')\bar{u}^{s'}(K')\gamma^{\mu}u^{s}(K)\bar{u}^{s}(K)\gamma^{\nu}\right] \quad (5.69)$$

Now we can use the completeness relation for the Dirac spinor

$$\sum_{s'} u^{s'}(K')\bar{u}^{s'}(K') = \not{K}' + M$$

to evaluate Equation 5.64 and we obtain

$$
\begin{aligned}
L_e^{\mu\nu} &= \frac{1}{2}\sum_{s,s'} T_r\left(u^{s'}(K')\bar{u}^{s'}(K')\gamma^\mu u^s(K)\bar{u}^s(K)\gamma^\nu\right) \\
&= \frac{1}{2}T_r\left[\left(\not{K}' + M\right)\gamma^\mu\left(\not{K} + M\right)\gamma^\nu\right]
\end{aligned}
\tag{5.70}
$$

The evaluation of the $L_e^{\mu\nu}$ is now reduced to an evaluation of traces of products of γ matrices. Several useful trace theorems as well as contraction theorems can be readily derived.

Trace Theorems:

$$
\begin{aligned}
&T_r\left(\text{odd number of } \gamma^\mu\right) = 0 \\
&T_r\left(\gamma^\mu\gamma^\nu\right) = 4g^{\mu\nu} \\
&T_r\left(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma\right) = 4\left(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda}\right)
\end{aligned}
\tag{5.71}
$$

Contraction Theorems:

$$
\begin{aligned}
\gamma_\mu\gamma^\mu &= 4 \\
\gamma_\mu\gamma^\nu\gamma^\mu &= -2\gamma^\nu \\
\gamma_\mu\gamma^\nu\gamma^\lambda\gamma^\mu &= 4g^{\nu\lambda} \\
\gamma_\mu\gamma^\nu\gamma^\lambda\gamma^\sigma\gamma^\mu &= -2\gamma^\sigma\gamma^\lambda\gamma^\nu
\end{aligned}
\tag{5.72}
$$

From Equation 5.71, the electron tensor becomes

$$
\begin{aligned}
L_e^{\mu\nu} &= \frac{1}{2}T_r\left(\not{K}'\gamma^\mu\not{K}\gamma^\nu\right) + \frac{1}{2}M^2 T_r\left(\gamma^\mu\gamma^\nu\right) \\
&= 2\left[K'^\mu K^\nu + K'^\nu K^\mu - \left(K\bullet K'\right)g^{\mu\nu} + M^2 g^{\mu\nu}\right]
\end{aligned}
\tag{5.73}
$$

Similarly, the muon tensor becomes

$$
L_e^{\mu\nu} = 2\left[P'_\mu P_\nu + P'_\nu P_\mu - \left(P\bullet P'\right)g_{\mu\nu} + M^2 g_{\mu\nu}\right] \tag{5.74}
$$

and finally

$$\overline{|M|^2} = \frac{8e^4}{q^4}\Big[(K'\bullet P')(K\bullet P) + (K'\bullet P)(K\bullet P') - M^2(P\bullet P') - M^2(K\bullet K') + 2M^2M^2\Big] \tag{5.75}$$

A useful relation for carrying out Lepton tensor contraction is

$$q^\mu L_{\mu\nu} = q^\nu L_{\mu\nu} = 0 \tag{5.76}$$

Equation 5.76 follows form current conservation since

$$\partial^\mu j_\mu = 0$$
$$\partial^\mu \left(\bar{u}(K')\gamma_\mu u(K) e^{-i(K-K')\bullet x}\right) = 0$$

Therefore

$$q^\mu \left(\bar{u}(K')\gamma_\mu u(K)\right) = 0$$

and

$$q^\mu L_{\mu\nu} = 0$$

Equation 5.76 can also be proven by noting

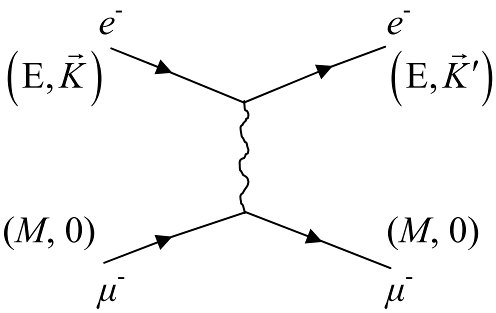
$$q^\mu \left(\bar{u}(K')\gamma_\mu u(K)\right) = \bar{u}(K')\not{q}\, u(K)$$
$$= \bar{u}(K')\left(\not{K} - \not{K}'\right)u(K) = 0$$
$$\text{(since } \left(\not{K} - m\right)u(K) = 0;\ \ \bar{u}(K')\left(\not{K}' - m\right) = 0\text{)}$$

Hence

$$\bar{u}(K')\left(\not{K} - \not{K}'\right)u(K) = \bar{u}(K')mu(K) - \bar{u}(K')mu(K) = 0$$

We consider three limiting cases for $e^-\mu^- \rightarrow e^-\mu^-$ scattering:

a) $M >> m$



For a very massive 'μ^-', there is no recoil, and the μ four-vector in the final state remains $(M, 0)$.

Also, $|\vec{K}'| = |\vec{K}|$

Note that in this case, C.M. frame is the same as lab frame.

Recall Equation 5.75

$$\overline{|M|^2} = \frac{8e^4}{q^4}\Big[(K'\bullet P')(K\bullet P) + (K'\bullet P)(K\bullet P') - M^2(P\bullet P') - M^2(K\bullet K') + 2M^2M^2\Big] \tag{5.75}$$

To evaluate Equation 5.75, we note

$$\begin{aligned}
&K'\bullet P' = K\bullet P = M\mathrm{E} \\
&K'\bullet P = K\bullet P' = M\mathrm{E} \\
&P\bullet P' = M^2 \\
&K\bullet K' = \mathrm{E}^2 - \vec{K}\bullet\vec{K}' = |K|^2 + M^2 - |K|^2\cos\theta = M^2 + 2|K|^2\sin^2\frac{\theta}{2} \\
&q^2 = (K-K')^2 = -(\vec{K}-\vec{K}')^2 = -2|K|^2(1-\cos\theta) = -4|K|^2\sin^2\frac{\theta}{2}
\end{aligned} \tag{5.77}$$

Equation 5.75 becomes

$$\overline{|M|^2} = \frac{8e^4}{16|K|^4 \sin^4 \frac{\theta}{2}} \Big[M^2\mathrm{E}^2 + M^2\mathrm{E}^2 - m^2 M^2 - M^2 m^2$$

$$- 2M^2 |K|^2 \sin^2 \frac{\theta}{2} + 2m^2 M^2 \Big] \tag{5.78}$$

$$= \frac{e^4}{|K|^4 \sin^4 \frac{\theta}{2}} \left[M^2\mathrm{E}^2 - M^2 |K|^2 \sin^2 \frac{\theta}{2} \right]$$

In the C.M. frame, the differential cross-section can be written as

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \overline{|m|^2} \frac{|K'|}{|K|} = \frac{\alpha^2 \mathrm{E}^2}{4|K|^4 \sin^4 \theta/2} \left(1 - v^2 \sin^2 \frac{\theta}{2} \right) \tag{5.79}$$

where we have used the following relations

$$s = (\mathrm{E} + M)^2 - K^2 = \mathrm{E}^2 - K^2 + M^2 + 2M\mathrm{E} = m^2 + M^2 + 2M\mathrm{E} \simeq M^2$$
$$|K| = vE$$
$$|K'| = |K|$$
$$\alpha = e^2/4\pi$$

Equation 5.79 is the Mott scattering formula, representing the scattering of a spin-½ charged particle off a static field.

Note that at the relativistic limit, $v \to 1$ and Equation 5.79 becomes

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 \mathrm{E}^2}{4|K|^4 \sin^4 \theta/2} \cos^2 \frac{\theta}{2} \tag{5.80}$$

In this case, electron is forbidden to scatter to 180° due to helicity conservation.

b) Muon can recoil, but E >> m and set $m = 0$

Equation 5.75 becomes

$$\overline{|M|^2} = \frac{8e^4}{q^4}\left[(K'\bullet P')(K\bullet P) + (K'\bullet P)(K\bullet P') - M^2(K\bullet K')\right]$$
$$= \frac{8e^4}{q^4}\left[(K\bullet K')(K\bullet P - K'\bullet P) + 2(K'\bullet P)(K\bullet P) - M^2(K\bullet K')\right]$$

where we have expressed P' as $P' = K + P - K'$, and used $K^2 = K'^2 = 0$.

Using

$$K\bullet K' = -q^2/2$$
$$P = (M,0)$$
$$q^2 = -4\mathrm{E}\mathrm{E}'\sin^2\theta/2$$
$$\nu = \mathrm{E} - \mathrm{E}' = -q^2/2M$$

we obtain

$$\overline{|M|^2} = \frac{8e^4}{q^4}\left[-q^2/2\,(\mathrm{E}M - \mathrm{E}'M) + 2(\mathrm{E}M)(\mathrm{E}'M) + 1/2\,M^2q^2\right]$$
$$= \frac{8e^4}{q^4}\left[2M^2\mathrm{E}\mathrm{E}'\right]\left[\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right] \tag{5.81}$$

Finally, we obtain

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4\mathrm{E}^2\sin^4\theta/2}\left(\frac{\mathrm{E}'}{\mathrm{E}}\right)\left[\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right] \tag{5.82}$$

(see pp. 131-132 of Halzen & Martin for the derivation of Equation 5.82 from Equation 5.81)

Note that the $\sin^2\frac{\theta}{2}$ term in Equation 5.82 allows the incident electron to scatter to 180o. This term is due to the magnetic moment of the muon, allowing spin-flip of the incident electron. This can be further illustrated by noting that for $e^-\pi^- \rightarrow e^-\pi^-$ scattering

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4\mathrm{E}^2 \sin^4 \theta/2}\left(\frac{\mathrm{E}'}{\mathrm{E}}\right)\cos^2\frac{\theta}{2} \tag{5.83}$$

<u>Note added:</u>

One can derive Equation 5.82 as follows:

Equations 5.33, 5.35, give

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \frac{\overline{|M|^2}}{4P_1 m_2}\;\frac{P_3}{16\pi^2\left[\mathrm{E}_1 + M_2 - \left(P_1\mathrm{E}_3/P_3\right)\cos\theta\right]}$$

$$\mathrm{E}_1 + M_2 - \left(P_1\mathrm{E}_3/P_3\right)\cos\theta = \mathrm{E}_1 + M_2 - \mathrm{E}_1\cos\theta$$

$$(\text{since } P_1 = \mathrm{E}_1,\ P_3 = \mathrm{E}_3 \text{ when } m_1 \rightarrow 0)$$

$$= \mathrm{E} + M - \mathrm{E}\cos\theta = 2\mathrm{E}\sin^2\frac{\theta}{2} + M$$

but

$$\nu = \mathrm{E} - \mathrm{E}' = -q^2/2M$$

and

$$q^2 = -4\mathrm{E}\mathrm{E}'\sin^2\theta/2$$

Therefore

$$2\mathrm{E}\sin^2\frac{\theta}{2} = \frac{-q^2}{2\mathrm{E}'} = \frac{2M\left(\mathrm{E}-\mathrm{E}'\right)}{2\mathrm{E}'} = M\left(\frac{\mathrm{E}}{\mathrm{E}'} - 1\right)$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \frac{\overline{|M|^2}}{4P_1 m_2}\;\frac{P_3}{16\pi^2\left(\mathrm{E}/\mathrm{E}' M\right)}$$

Using Equation 5.81, we obtain

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \frac{\alpha^2}{4E^2\sin^4\theta/2}\left(\frac{E'}{E}\right)\left[\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right]$$

c) Relativistic Limit (neglecting both m^2 and M^2)

In this limit, Equation 5.75 simplifies to

$$\overline{|M|^2} = \frac{8e^4}{q^4}\left[\left(K'\bullet P'\right)\left(K\bullet P\right) + \left(K'\bullet P\right)\left(K\bullet P'\right)\right] \qquad (5.81)$$

Neglecting m^2 and M^2, the Mandelstam variables become

$$\begin{aligned} s &= \left(K + P\right)^2 \simeq 2K\bullet P = 2K'\bullet P' \\ t &= \left(K - K'\right)^2 \simeq -2K\bullet K' = -2P\bullet P' \\ u &= \left(K - P'\right)^2 \simeq -2K\bullet P' = -2K'\bullet P \end{aligned}$$

Equation 5.81 becomes

$$\overline{|M|^2} = 2e^4\,\frac{s^2 + u^2}{t^2} \qquad (5.82)$$

and the C.M. cross-section is

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{1}{64\pi^2 s}\frac{P_f}{P_i}\overline{|M|^2} = \frac{\alpha^2}{2s}\left(\frac{1+\cos^4 \theta/2}{\sin^4 \theta/2}\right) \qquad (5.83)$$

where we use the following expressions in the C.M. frame:

$s = 4K^2$ $\qquad$ $t = -2K^2\,(1 - \cos\theta)$ $\qquad$ $u = -2K^2\,(1 + \cos\theta)$

Although one cannot check the expression for $e^-\mu^- \rightarrow e^-\mu^-$ scattering at the high energy limit, one can consider several related reactions which can be and have been studied experimentally.

c.1) $e^- e^{\pm} \to \mu^- \mu^{\pm}$

The scattering amplitude for this process can be obtained from $e^- \mu^- \to e^- \mu^-$ by interchanging $s \leftrightarrow t$ first, giving $e^- e^+ \to \mu^- \mu^+$, followed by $t \leftrightarrow u$ exchange, resulting in $e^- e^+ \to \mu^- \mu^+$.

Hence from Equation 5.82, one obtains for $e^- e^+ \to \mu^- \mu^+$

$$\overline{|M|^2} = 2e^4 \frac{u^2 + t^2}{s^2} \tag{5.84}$$

The differential cross-section for $e^- e^+ \to \mu^- \mu^+$ is

$$\left(\frac{d\sigma}{d\Omega}\right)_{e^-e^+ \to \mu^-\mu^+} = \frac{\alpha^2}{2s} \frac{4K^4\left(2 + 2\cos^2\theta\right)}{16K^4} = \frac{\alpha^2}{4s} \left(1 + \cos^2\theta\right) \tag{5.85}$$

and the total cross-section is

$$\sigma\left(e^- e^+ \to \mu^- \mu^+\right) = \frac{4\pi\alpha^2}{3s} \tag{5.86}$$

Both the $(1 + \cos^2\theta)$ angular distribution in Equation 5.85 and the $\frac{1}{s}$ dependence of the total cross-section in Equation 5.86 are well confirmed by experiments.

c.2) $e^- e^+ \to q\overline{q}$

This process is analogous to the $e^- e^+ \to \mu^- \mu^+$ scattering. An important difference, apart from the factor Q_q^2 for the quark charge, is the color factor of 3 to account for the 3 colors for the quarks.

$$\begin{aligned} \sigma\left(e^- e^+ \to q\overline{q}\right) &= \frac{4\pi\alpha^2}{3s} \times Q_q^2 \times 3 \\ &= 3 \times Q_q^2 \; \sigma\left(e^- e^2 \to \mu^- \mu^+\right) \end{aligned} \tag{5.87}$$

Experimentally, quark-antiquark pairs are not observed. Instead, the hadrons into which the $q\overline{q}$ hadronize are detected. One can measure the R factor, defined as

$$R = \frac{\sigma\left(e^- e^+ \to \text{hadrons}\right)}{\sigma\left(e^- e^+ \to \mu^- \mu^+\right)} = 3\sum_q Q_q^2 \tag{5.88}$$

Depending on the C.M. energy, various $q\overline{q}$ channels may be open. One expects R to be:

$$R = 3\left[\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2\right] = 2$$

if u, d, s quark pairs can be produced.

If the C.M. is above the threshold for charm-quark pair production, then

$$R = 2 + 3\left(\frac{2}{3}\right)^2 = \frac{10}{3}$$

and (5.89)

$$R = \frac{10}{3} + 3\left(\frac{1}{3}\right)^2 = \frac{11}{3}$$

once the $b\overline{b}$ threshold is passed.

The experimental data are in good agreement with the expectations from Equation 5.89.

Note that if there is no color factor of 3, the predicted R would be in a strong disagreement with the data.

c.3) $q\overline{q} \to e^- e^+$, $q\overline{q} \to \mu^- \mu^+$

This is the inverse reaction of $e^- e^+ \to q\overline{q}$. It can be studied experimentally in hadron-hadron interaction, in which a quark from one hadron interacts with the antiquark in the other hadron. This process is also called the Drell-Yan process.

The cross-section for this process is analogous to the $e^-e^+ \rightarrow q\overline{q}$, with an important difference. The color degree of freedom for the quarks / antiquarks implies that only $q-\overline{q}$ with matched color (blue-antiblue, for example) can annihilate. Hence the cross-section is

$$\sigma\left(q\overline{q} \rightarrow e^-e^+\right) = \sigma\left(e^-e^+ \rightarrow \mu^-\mu^+\right)\frac{Q_q^2}{3} \tag{5.90}$$

Also, the angular distributions show a $1 + \cos^2\theta$ dependence. Both the cross-section and the $1 + \cos^2\theta$ dependence have been verified experimentally.