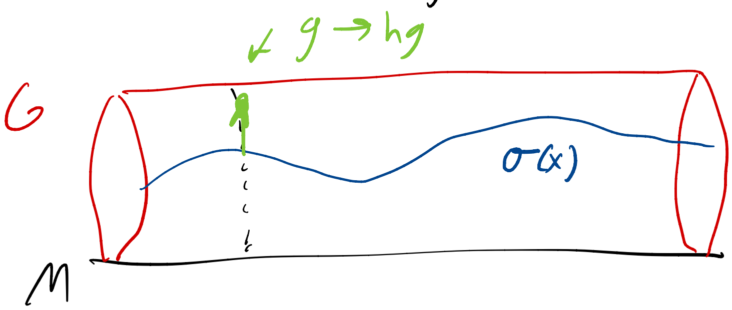
Geometry of gauge theory

When you first encountered gauge theory, you may have wondered why $\partial_\mu + igA_\mu$ was called a "covariant" derivative, why A_μ had a funny inhomogeneous transformation rule, and why we say "only the fields are physical, except in situations like the Aharonov-Bohm effect." We will see today how all these facts are related to differential geometry objects defined on a principal bundle.

Definitions: $P \xrightarrow{G} M$ is a principal G-bundle over M if it looks locally like (x, g) where $x \in M$ is a point on a differentiable (for us, Lorentzian) manifold M and $g \in G$ is an element of a Lie group. M is the "base space" and G is the gauge group.



Example: $M = \mathbb{R}$, $G = U(1) \simeq S^1$. Then P is the cylinder.

A map $\sigma : M \to P$ which can be projected back down to M is a section: a group element at each point. This is the local gauge transformation you have seen before. So far, the group can only act on itself by moving points along a fiber (copy of G over a point x), later we will see how G acts on other geometric objects on M.

To make the geometric analogies clearer, let's review some facts about covariant derivatives and connections on Lorentzian manifolds

The covariant derivative lets us compare vectors defined at different points of M. In other words, it induces a map $T_x M \to T_y M$ (parallel transport), so we can talk about a connection on <u>the tangent bundle</u>.

In local coordinates, the connection is described by Christoffel symbols $\Gamma^\rho_{\mu\sigma}$, and the covariant derivative of a vector field V is $\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda$. You may recall that under a general coordinate transformation, Γ does <u>not</u> transform like a tensor:

$$\Gamma \longrightarrow \left(\frac{\partial x}{\partial x'}\right)^2 \left(\frac{\partial x'}{\partial x}\right)\Gamma - \underbrace{\left(\frac{\partial x}{\partial x'}\right)^2 \left(\frac{\partial^2 x'}{\partial x^2}\right)}_{\text{extra non-tensorial piece}}$$

In terms of Γ, the Riemann tensor is

$$R \text{ "=" } \partial_{[}\Gamma_{]} + [\Gamma, \Gamma]$$

and we also have $[\nabla, \nabla] = R$ for a torsion-free connection.

We have written this in such a way as to remind you of various formulas from gauge theory:

$$A_\mu \longrightarrow A_\mu + \partial_\mu \alpha \quad \text{(inhomogeneous under gauge transformations)}$$

$$F_{\mu\nu} = [D_\mu, D_\nu] = \partial_{[\mu} A_{\nu]} - ig[A_\mu, A_\nu]$$

Indeed, these are the <u>same</u> geometric objects! A is a connection, F is a curvature, etc., but they are defined on P instead of M.

Make this more precise by introducing the connection 1-form.

First, define a vielbein $\hat{e}_a = e_a{}^\mu \partial_\mu$, and a dual basis of 1-forms $\hat{\theta}^a = e^a{}_\mu dx^\mu$, such that the metric is diagonal:

$$g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu = \eta_{ab}\, \hat{\theta}^a \otimes \hat{\theta}^b .$$

Using the vielbein, we can define connection coefficients Γ^a_{bc} by $\nabla_a \hat{e}_b = \Gamma^c_{ab}\, \hat{e}_c$ as usual, and from these, a connection 1-form (also called the "spin connection"):

$$\omega^a{}_b \equiv \Gamma^a_{bc}\, \hat{\theta}^c \equiv \omega_\mu{}^a{}_b\, dx^\mu \quad \text{where} \quad \omega_\mu{}^a{}_b = e_\mu^c\, \Gamma^a_{bc}.$$

all vielbein indices

one coordinate index

Two important properties of ω^a_b will be important for us:

- for a metric connection, $\nabla g = 0$, we have

$$0 = \nabla_\mu \eta_{ab} = \cancel{\partial_\mu \eta_{ab}} - \omega_\mu{}^c{}_a \eta_{cb} - \omega_\mu{}^c{}_b \eta_{ac}$$

$$= -\omega_{\mu ab} - \omega_{\mu ba}$$

In other words, ω is asymmetric in the vielbein indices.

We can interpret $\omega_\mu{}^a{}_b$ as $(\omega_\mu)^a{}_b$, a matrix-valued 1-form, where the matrix $\omega^a{}_b$ is in the Lie algebra of local Lorentz, $\mathfrak{so}(3,1)$.

- Under a change of frame which preserves the metric, vielbein changes by a Lorentz transformation, $e^a{}_\mu \to \Lambda^a{}_b\, e^b{}_\mu$, and similarly lower vielbein indices have $e^\mu{}_a \to (\Lambda^{-1})^b{}_a\, e^\mu{}_b$. We can do this at each point: $\Lambda^a_b(x)$. The connection coefficients will transform as

$$\nabla_{a'} \hat{e}'_b \equiv \Gamma'^c_{ab}\, \hat{e}'_c = \underbrace{\nabla_{a'}\left((\Lambda^{-1})^c_b\, \hat{e}_c\right)}_{\text{inhomogeneous!}} + (\Lambda^{-1})^c_b\, \underbrace{\nabla_{a'} \hat{e}_c}_{\propto\, \Gamma}$$

Keeping careful track of the indices, we find the transformation

$$\omega' = \Lambda^{-1} \omega \Lambda + \Lambda^{-1} d \Lambda$$

Not coincidentally, this looks a lot like a gauge transformation! And converting back to coordinate basis, it gives the familiar non-tensorial transformation of the Christoffels.

Geometrically, what is happening is that there is a principal bundle over M, the <u>frame bundle</u> $P \equiv LM \xrightarrow{G} M$, where $G = SO(3,1)$, which comes with every Lorentzian manifold M. The local Lorentz transformations $\Lambda(x)$ are a <u>section</u> $\sigma : M \to P$. We can even go one step further and define a connection on the (target space of the) <u>total space</u> which is independent of σ. In local coordinates where a point on P is $p = (x, g)$ with $g = \sigma(x)$,

$$\hat{\omega} = g^{-1} \omega g + g^{-1} dg.$$

A different section $\sigma'(x)$ is related to σ by $\sigma'(x) = hg$. To get back to P, need to left-multiply by h^{-1}. The new connection is

$$\hat{\omega}' = (h^{-1}g)^{-1} \omega' (h^{-1}g) + (h^{-1}g)^{-1} d(h^{-1}g)$$

$$= g^{-1} h (h^{-1} \omega h + h^{-1} dh)(h^{-1} g) + (g^{-1} h)(-h^{-1} dh\, h^{-1} g + h^{-1} dg)$$

$$= g^{-1} \omega g + g^{-1} \cancel{dh\, h^{-1}} g - g^{-1} \cancel{dh\, h^{-1}} + g^{-1} dg$$

$$= \hat{\omega}, \text{ so independent of } \sigma \text{ as desired.}$$

In other words, $\hat{\omega}$ (which lives on P) is a geometric object independent of the choice of section (gauge), and pulling it down to the base yields a 1-form on M which <u>does</u> transform under a choice of gauge.

Now, it is a straightforward conceptual jump to take G to be an arbitrary Lie group instead of local Lorentz. In this case, there is no frame bundle but we can still define a $\mathfrak{g}$-valued 1-form $\hat{\omega}$ on P, choose a section, and get a $\mathfrak{g}$-valued connection 1-form ω on the base. Reversing the logic, asking for invariance of $\hat{\omega}$ implies the gauge transformation rule for ω, which we can now call iA. For $G = U(1)$, writing $g = e^{i\alpha}$ gives

$$iA \to g^{-1} iA g + g^{-1} dg = e^{-i\alpha} iA e^{i\alpha} + e^{-i\alpha}(i d\alpha\, e^{i\alpha})$$
$$= iA + i d\alpha, \text{ so}$$

$A \to A + d\alpha$, familiar from E+M.

Just like the covariant derivative on TM is $\nabla = \partial + \omega$, the covariant derivative on TP is $D = \partial + \hat{\omega}$, which when pulled back to M by a gauge choice σ is $D = \partial + A$.

Also get a nice interpretation of curvature: on M, can show $R = d\omega + \omega \wedge \omega$ is the curvature 2-form which is matrix-valued and whose components are the Riemann tensor. On P, the analogous object $\Omega = d\hat{\omega} + \frac{1}{2}[\hat{\omega} \wedge \hat{\omega}]$ can be interpreted as the curvature of P with respect to the connection $\hat{\omega}$. Pulled back to M, this becomes the familiar field strength tensor (antisymmetric, so really a 2-form) $F = dA + A \wedge A$, which transforms as $F \to g^{-1} F g$.