QED with quarks

We will now augment the QED Lagrangian with the remaining fermions,

$$\mathcal{L} \supset \sum_{f=1}^{3} Q_f^\dagger \bar{\sigma}^\mu D_\mu Q_f + u_R^{f\dagger} \sigma^\mu D_\mu u_R^f + d_R^{f\dagger} \sigma^\mu D_\mu d_R^f - y_{ij}^d Q_i^\dagger H d_{Rj} - y_{ij}^u Q_i^\dagger \tilde{H} u_{Rj}$$

Just like in QED, where $H \to \begin{pmatrix} 0 \\ v \end{pmatrix}$ and leptons got mass and electric charge, same thing happens for quarks:

$$y_{ij}^d Q_i^\dagger H d_{Rj} \longrightarrow m_{d_f} d_L^{f\dagger} d_R^f$$

$$y_{ij}^u Q_i^\dagger \tilde{H} u_{Rj} \longrightarrow m_{u_f} u_L^{f\dagger} u_R^f$$

Recall hypercharges: $Y = \frac{1}{6}$ for Q, $Y = \frac{2}{3}$ for u_R, $Y = -\frac{1}{3}$ for d_R

Electric charge is $T_3 + Y = \begin{cases} \frac{1}{2} + \frac{1}{6} = \frac{2}{3}, & u_L \\ -\frac{1}{2} + \frac{1}{6} = -\frac{1}{3}, & d_L \\ 0 + \frac{2}{3} = \frac{2}{3}, & u_R \\ 0 + \left(-\frac{1}{3}\right) = -\frac{1}{3}, & d_R \end{cases}$

$\Rightarrow$ in the Standard Model, up-type quarks are charge $\frac{2}{3}$ fermions, down-type quarks are charge $-\frac{1}{3}$. We will describe experiments which test <u>both</u> spin and charge.

Note: quarks also interact with $SU(3)_c$ gauge field; we will add this back in shortly.

$$\Rightarrow \mathcal{L}_{quarks} = \sum_{f=1}^{3} \left(\bar{u}_f (i\partial\!\!\!/ + \tfrac{2}{3} e A\!\!\!/) u_f + \bar{d}_f (i\partial\!\!\!/ - \tfrac{1}{3} e A\!\!\!/) d_f - m_{u_f} \bar{u}_f u_f - m_{d_f} \bar{d}_f d_f \right)$$

Only new Feynman rule is factor of $\frac{2}{3}$ or $\frac{-1}{3}$ on quark-quark-photon vertex.

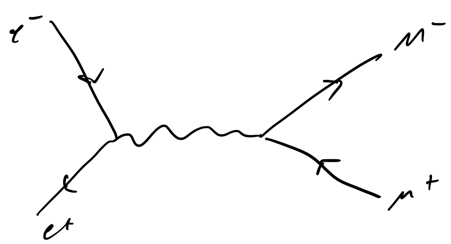
Let's use QED to test the predicted properties of quarks.

A first glimpse of quarks: $e^+e^- \to$ hadrons.

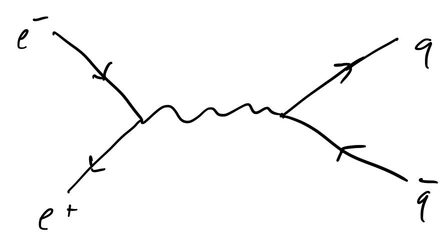
Nomenclature reminder: "hadrons" = any strongly-interacting particles. Pions, kaons, protons, neutrons, ... these are what are <u>actually observed</u> in experiments. Free quarks are not observed! (More on this in PHYS 570 and next lecture)

We will compute $R = \dfrac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$ as a function of $\sqrt{s} = E_{cm}$, approximating the numerator by $\sigma(e^+e^- \to q\bar{q})$.

Not obvious this should work; isn't cross section strongly affected by strong interactions among quarks? Will justify this shortly.



vs.



In limit where all particles are massless, these diagrams are identical up to $e \to Q_i e$. $\frac{d\sigma}{d\cos\theta} \sim 1 + \cos^2\theta$, just like $\mu^+\mu^-$!

Experimental confirmation that quarks are spin-½.

$$\Rightarrow \sigma(e^+e^- \to \text{all quarks}) = 3 \times \sum_{m_i < \sqrt{s}/2} Q_i^2 \, \sigma(e^+e^- \to \mu^+\mu^-)$$

quarks are a 3-component vector under SU(3)

need enough energy to make 2 quarks

$m_u \simeq 2$ MeV, $m_d \simeq 5$ MeV, $m_s \simeq 100$ MeV, but $m_c \simeq 1.5$ GeV, so for $\sqrt{s} \simeq$ GeV, not enough energy to produce $c\bar{c}$

$$\Rightarrow R(\sqrt{s} = 1\text{ GeV}) = 3\left(\left(\tfrac{2}{3}\right)^2 + \left(-\tfrac{1}{3}\right)^2 + \left(-\tfrac{1}{3}\right)^2\right) = 2$$

$q=u$ $\quad q=d \quad$ $q=s$

Well-matched by experiment! Experimental confirmation that quarks have 3 colors, <u>and</u> that quarks have fractional charges.

QCD at colliders

Add back in two more terms from the SM Lagrangian

$$\mathcal{L} \supset -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu a} + \sum_{i,j=1}^{N} \sum_f \bar{\Psi}^f_i (\delta_{ij}\, i\not{\partial} + g_s \not{A}^a T^a_{ij} - m_f \delta_{ij}) \psi^f_j$$

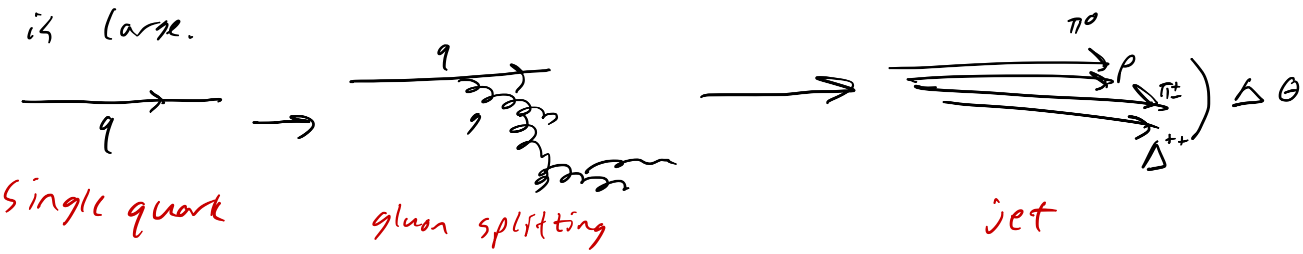
$$// \quad -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc} A^b_\mu A^c_\nu)^2$$

; A^a_μ is the gluon field

The crucial difference between QED and QCD is the gluon self-interaction.

This leads to interesting phenomena:

- Asymptotic freedom. At high energies, the strong force coupling g_s gets weaker. This means we can borrow many of our results from QED and tack on some group theory factors to get the right answer.
- At lower energies, gluons make more gluons, and the interaction strength is large.



Instead of free quarks, what we see at colliders is a spray of nearly-collimated hadrons, called a jet. The evolution from quark to jet is calculable as long as $\alpha_s < 1$; more on this in PHYS 570.

- At an energy of about 200 MeV, $\alpha_s \equiv \frac{g_s^2}{4\pi} = 1$, so perturbation theory based on Feynman diagrams breaks down. Two options for calculating in a nonperturbative field theory:
  - discretize spacetime on a finite lattice and use a computer (lattice gauge theory) ← Prof. El-Khadra does this
  - use symmetry arguments to find a change of variables to describe some subset of the particles at low energy (chiral perturbation theory) ← we will briefly do this next week

We will focus on the high-energy part of this sequence in this course, leaving the lower-energy phenomena for PHYS 570.

Group theory review

First we review some group theory facts about $SU(N)$ where $N=3$.

- $SU(3)$ is 8-dimensional: $U^\dagger U = \mathbb{1}$ enforces 9 algebraic constraints on 9 complex (18 real) numbers; requiring $\det U = 1$ enforces one more.
- By writing $U = \mathbb{1} + iX$, we find $(\mathbb{1} - iX^\dagger)(\mathbb{1} + iX) = \mathbb{1} \Rightarrow X^\dagger = X + \mathcal{O}(X^2)$
Similarly, $\det U = \mathbb{1} \Rightarrow \mathrm{Tr}(X) = 0$ (we showed this in week 3).
So Lie algebra $\mathfrak{su}(3)$ is traceless Hermitian 3×3 matrices.
Conventional to choose the generators $T^a = \frac{1}{2}\lambda^a$, $a = 1, \ldots 8$, where λ^a are the Gell-Mann matrices (see Schwartz (25.17)).
- The structure constants of $\mathfrak{su}(3)$ are defined by $[T^a, T^b] = i f^{abc} T^c$.
- Just like for $SU(2)$ and $SO(3,1)$, there are multiple representations of the group. There is a very neat mathematical generalization of the raising/lowering operator trick to find these representations, but we will focus on the two that exist for any $SU(N)$: fundamental (dim. $N = 3$) and adjoint (dim. $N^2 - 1 = 8$)
- The fundamental rep is straightforward: $(T^a_F)_{ij} = \frac{1}{2}\lambda^a_{ij}$. The generators are 3×3 matrices, and they satisfy
$\mathrm{Tr}(T^a_F T^b_F) \equiv T^a_{ij} T^b_{ji} = \frac{1}{2}\delta^{ab}$. For Lie algebras, taking the trace acts like an inner product (for math nerds, this is known as the Killing form). The coefficient is $T_F = \frac{1}{2}$. We can also sum over generators:
$$\sum_a (T^a_F T^a_F)_{ij} = C_F \delta_{ij}, \text{ where } C_F = \frac{N^2-1}{2N} = \frac{4}{3}$$ is the quadratic Casimir in the fundamental representation. Exactly analogous to $J^2 = \sum J^i J^i = s(s+1)\mathbb{1}$ for spin $SU(2)$. Quarks are vectors in the fundamental representation, and transform as
$\psi_i \rightarrow \psi_i + i\alpha^a (T^a_F)_{ij} \psi_j$. Antiquarks ($\psi^\dagger$ or $\bar{\psi}$) transform as
$\bar{\psi}_i \rightarrow \bar{\psi}_i - i\alpha^a \bar{\psi}_j (T^a_F)_{ji}$ (Note: Q, u_R, d_R are all in the same representation, which is why we can use 4-component spinors which combine u_L and u_R)

- The adjoint rep. is a representation of the Lie algebra on itself. (This sounds weird and mysterious the first time you hear it, but it's the simplest way of stating it.)

What is a representation? $V \xrightarrow{T} V'$, meaning a vector V gets mapped to a vector V' under a Lie algebra element T. But this is precisely what the commutation relations do!

$$T^a \xrightarrow{T^b} i f^{abc} T^c, \text{ where the map is } [T^a, T^b].$$

Because T^c is a linear combination of the other generators, we must be able to write this map as an 8×8 matrix $(T^a_{adj})_{bc}$, whose entries are $(T^a_{adj.})_{bc} = i f^{bac}$.

The inner product for the adjoint is $\mathrm{Tr}(T^a_{adj.} T^b_{adj.}) = \sum f^{acd} f^{bcd} = N\delta^{ab}$

The quadratic Casimir is $\sum_a (T^a_{adj} T^a_{adj.})_{bc} = -\sum f^{bad} f^{dac} = \sum f^{bad} f^{cad} = N\delta^{bc}$,

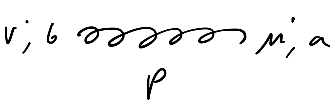
so $T_A = C_A = 3$.

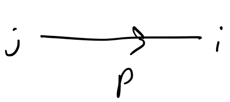
Gluons are vectors in the adjoint representation:

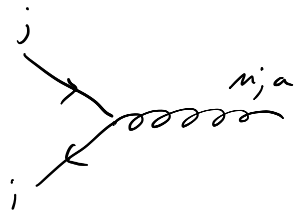
$$A^b_\mu \longrightarrow A^b_\mu + i\alpha^a (T^a_{adj.})_{bc} A^c_\mu + \frac{1}{g_s}\partial_\mu \alpha^b$$

$$\Longleftrightarrow A^a_\mu \longrightarrow A^a_\mu - f^{abc}\alpha^b A^c_\mu + \frac{1}{g_s}\partial_\mu \alpha^a$$

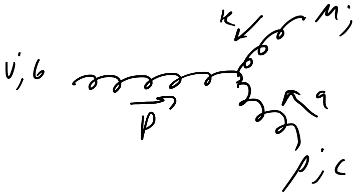
With this group theory technology, we can now write down the Feynman rules for QCD:


$= \dfrac{-i\eta^{\mu\nu}}{p^2}\delta^{ab}$ (gluon is just like photon with a δ^{ab} for color in adjoint rep.)

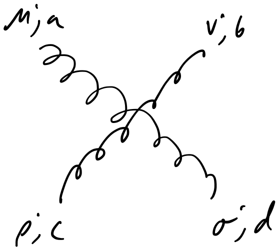

$= \dfrac{i(\not{p}+m)}{p^2-m^2}\delta^{ij}$ (quarks are just like electrons with δ^{ij} for color in fundamental rep.)


$= i g_s \gamma^\mu T^a_{ij}$ (order matters because T^a_{ij} is a matrix!)

So far, so good... now comes the mess.



$$= g_s f^{abc}\left[\eta^{\mu\nu}(k-p)^\rho + \eta^{\nu\rho}(p-q)^\mu + \eta^{\rho\mu}(q-k)^\nu\right]$$



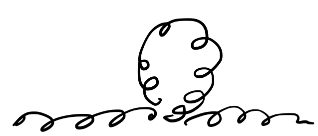
$$= -ig_s^2\left[f^{abe}f^{cde}\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}\right) + (2 \text{ permutations})\right]$$

Even computing $gg \to gg$ requires 1000 terms! We will not do this in this class, but there is a beautiful mathematical formalism which simplifies things enormously (see Schwartz Ch. 27 if you're curious).

Asymptotic Freedom

In QED, 1-loop diagrams like 

lead to vacuum polarization. Just like a dielectric screens electric charge at long distances, virtual e^+/e^- pairs screen the coupling e, such that $\mu \frac{d}{d\mu} e = \frac{e^3}{12\pi^2}$, where μ is an energy scale. The RHS is known as the beta function of QED, and because it is positive, e increases with increasing μ. $e(\mu)$ is not a constant but runs with energy.

In QCD, the opposite happens. Diagrams like  lead to anti-screening, such that

$$\mu \frac{d}{d\mu} g_s = -\frac{g_s^3}{16\pi^2}\left[\frac{11}{3}C_A - \frac{4}{3}n_f T_F\right].$$ (Nobel prize 2004!)

For $SU(3)$ with six quark flavors, $n_f = 6$, $C_A = 3$, $T_F = \frac{1}{2}$, so RHS is

$$\frac{-g_s^3}{16\pi^2}\left(\frac{11}{3}(3) - \frac{4}{3}\left(\frac{1}{2}\right)(6)\right) = -\frac{7g_s^3}{16\pi^2} < 0,$$ so g_s decreases as μ increases. This is known as asymptotic freedom, and is why we can approximate quarks as weakly-interacting and use perturbative QFT at high energies, where $\alpha_s \simeq 0.1$. (This is why our R-ratio approximation worked.)