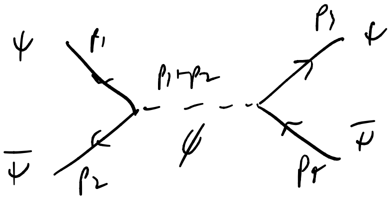
Introduction to effective field theories

So far in this course we have considered almost exclusively dimension-4 operators. There is a good reason for this: in QFT, field theories with scalars, fermions, and gauge bosons with interactions up to dimension 4 are <u>renormalizable</u>, meaning that any apparently infinite quantities from loop diagrams can be consistently removed. However, this is <u>not</u> the case for higher-dimension operators, which are called non-renormalizable. In general, field theories with these operators require an infinite series of ever-higher dimension operators to cancel would-be infinities. The coefficients of these operators are related to measurable quantities, so these theories are still predictive.

We saw an example in the 4-Fermi theory of how a renormalizable Lagrangian at high energies gives a non-renormalizable one at low energies. Let's make that precise with a toy example:

$$\mathcal{L} = i\bar{\psi}\partial_\mu\psi - m\bar{\psi}\psi + \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}M^2\phi^2 - y\,\phi\bar{\psi}\psi.$$

This describes a fermion of mass m interacting with a scalar of mass M through a Yukawa coupling. Consider $\psi\bar{\psi}$ scattering:



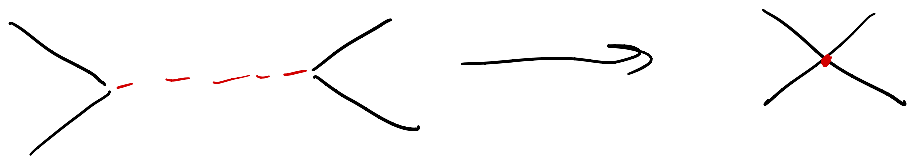
$$i\mathcal{M} = (-iy)^2\left(\frac{i}{s-M^2}\right)\bar{v}(p_2)u(p_1)\,\bar{u}(p_3)v(p_4)$$

Suppose scattering takes place at center-of-mass energies $\sqrt{s} \ll M$. Then we can expand the amplitude using

$$\frac{1}{s-M^2} = \frac{-1}{M^2}\,\frac{1}{1-\frac{s}{M^2}} = -\frac{1}{M^2}\left(1+\frac{s}{M^2}+\frac{s^2}{M^4}+\ldots\right)$$

So $$i\mathcal{M} = iy^2\,\bar{v}(p_2)u(p_1)\,\bar{u}(p_3)v(p_4)\left[\frac{1}{M^2}+\frac{s}{M^4}+\frac{s^2}{M^6}+\ldots\right]$$

The first term looks like a 4-fermion interaction with coefficient $\frac{y^2}{M^2}$: $\mathcal{L}_{eff} \supset \frac{y^2}{M^2}\bar{\Psi}\Psi\bar{\Psi}\Psi$. The interpretation is that at very low energies, much less than M, the ϕ particle cannot be produced on-shell. The amplitude for its propagation becomes very small the farther off-shell it is, so the propagator in the original matrix element shrinks to a point:



However, this is just the leading-order contribution. The other terms in the expansion represent Lagrangian terms like

$$\mathcal{L} \supset \frac{y^2}{M^4}\partial_\mu\bar{\Psi}\partial^\mu\Psi\bar{\Psi}\Psi + \frac{y^2}{M^6}\partial_\mu\bar{\Psi}\partial_\nu\Psi\partial^\mu\bar{\Psi}\partial^\nu\Psi + \ldots$$

with increasing numbers of derivatives, which become factors of momenta in the Feynman rules. We say that we have integrated out the ϕ particle and encapsulated its effects in an infinite series of operators containing only Ψ.

Another perspective: the equation of motion for ϕ is $(\Box + M^2)\phi = -y\bar{\Psi}\Psi$. We can "solve" for ϕ as a formal power series:

$$\phi = \frac{-1}{\Box + M^2}(y\bar{\Psi}\Psi).$$

By replacing ϕ with its solution in the equations of motion for Ψ and expanding for small momenta we obtain the same series of operators we got before:

$$\mathcal{L} \supset -y\bar{\Psi}\Psi\phi \longrightarrow -y^2\bar{\Psi}\Psi\left(\frac{1}{M^2} - \frac{\Box}{M^4} + \ldots\right)\bar{\Psi}\Psi$$

This manipulation (replacing a field with its classical solution) is also referred to as integrating out a field. (In path integral formalism in QFT, we compute the path integral exactly over the heavy field). In fact, nothing here needs quantum mechanics (yet!), this works perfectly fine for classical nonlinear fields like GR.

We can also run this procedure in reverse: for a given higher-dimensional operator, what additional heavy particles would we have to add to the theory to give our desired operator once they are integrated out? This is known as a UV completion, and is in general not unique, though it can point to where to look for new physics responsible for that operator. Non-renormalizable theories predict their own demise!

ex. Chiral Lagrangian $\xrightarrow{\text{UV complete}}$ QCD

$$\mathcal{L} = F_\pi^2 \, \mathrm{Tr}(D_\mu U D^\mu U^\dagger) \qquad \Lambda_{QCD} \gtrsim 200 \text{ MeV} \qquad \mathcal{L} = -\tfrac{1}{4} G^a_{\mu\nu} G^{\mu\nu a} + \bar{\psi} \not{D}_\mu \psi$$

Violates unitarity at $\sqrt{s} \sim \sqrt{4\pi} F_\pi \gtrsim 400$ MeV

Becomes non-perturbative at $\Lambda_{QCD} \gtrsim 200$ MeV

We can systematically account for new physics at high energy scales with the Standard Model Effective Field Theory (SMEFT): all $SU(3) \times SU(2) \times U(1)$-invariant operators of any dimension built out of SM fields

$$\mathcal{L}_{SMEFT} = \sum_{d=5}^{\infty} \frac{1}{\Lambda^{d-4}} \sum_i^{N^{(d)}} C_i^{(d)} \mathcal{O}_i^{(d)}$$

Λ: mass scale where UV completion is required

$C_i^{(d)}$: Wilson coefficients

$\mathcal{O}_i^{(d)}$: operators of mass dimension d

Any measurement of nonzero C_i is by construction proof of physics beyond the SM! However, combinatorics and avoiding double-counting is very tricky.

$d=5$: $N^{(5)} = 2$ (Weinberg operator and its conjugate)

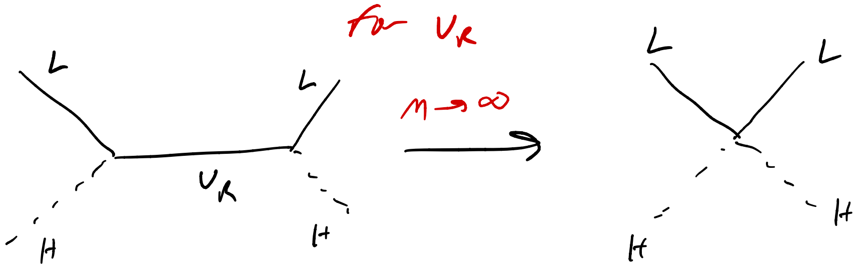
$N^{(6)} = 84$, $N^{(7)} = 30$, $N^{(8)} = 993$, ... (see arxiv:1512.03433 if you're curious)

Some examples:

- Weinberg operator $\mathcal{O}^{(5)} = \frac{1}{\Lambda} \epsilon^{\alpha\beta} (\epsilon^{ab} \underset{Y=-\frac{1}{2}}{L_{a\alpha}} \underset{Y=\frac{1}{2}}{H_b})(\epsilon^{cd} L_{c\beta} H_d) + h.c.$

can be UV-completed with a heavy right-handed neutrino ν_R:

$$\mathcal{L} = Y_\nu L^+ \tilde{H} \nu_R - \underbrace{\frac{M}{2} \epsilon^{\alpha\beta} \nu_{R\alpha} \nu_{R\beta}}_{\text{Majorana mass for } \nu_R}$$



ν_R propagator is $\frac{\not{p} + M}{p^2 - M^2} = \frac{-1}{M} + \mathcal{O}\left(\frac{p}{M}\right)$

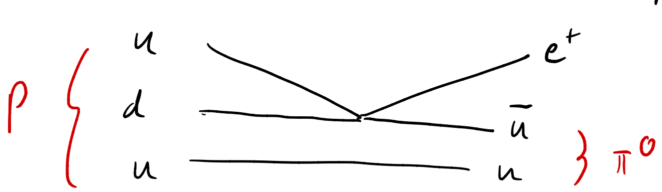
$\Rightarrow$ identify $\frac{1}{\Lambda} \equiv \frac{1}{M}$, $C^{(5)} = (Y_\nu)^2$

If neutrinos get mass from the Weinberg operator, the value of the mass suggests a mass for a new heavy right-handed neutrino: the lighter the SM neutrinos, the heavier ν_R is ("seesaw mechanism").

(Recall from HW 3 $m_\nu \sim \frac{v^2}{\Lambda}$; $m_\nu < 0.9\,eV$ from cosmology $\Rightarrow \Lambda \gtrsim 10^{15}\,GeV$)

- Proton decay. In the SM, protons are absolutely stable because they are the lightest baryon, and baryon number is conserved. But baryon number is an accidental symmetry, and is generically violated in the SMEFT.

Consider $\mathcal{O}^{(6)} = \frac{1}{\Lambda_6^2} \epsilon^{ijk} Q_i Q_j Q_k L$, where ϵ^{ijk} is the color antisymmetric tensor and all SU(2) and fermion indices are contracted with the appropriate $\epsilon^{\alpha\beta}$. This leads to:



$p \to e^+ \pi^0$

Experiments such as Super-Kamiokande have been searching for this decay for decades: all null results so far!

$\tau_p > 1.67 \times 10^{34}$ yr from $\pi^0 e^+$ channel.

$\Rightarrow \Gamma_{p \to \pi^0 e^+} < 1.2 \times 10^{-57}$ eV

Let's use this to bound Λ_6:

$$\langle |M|^2 \rangle \simeq \frac{1}{\Lambda_6^4} \times [E]^6$$

to calculate this exactly requires non-perturbative QCD: the largest scale in the problem is m_p, so let's just set $E = m_p$.

$$\Gamma \simeq \frac{1}{2m_p} \frac{1}{8\pi} \frac{m_p^6}{\Lambda_6^4} \simeq \frac{m_p^5}{16\pi \Lambda_6^4}$$

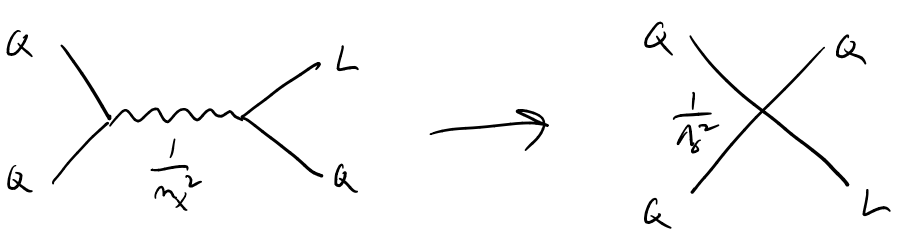
$\Rightarrow \Lambda_6 > 1.0 \times 10^{16}$ GeV!!

What physics could possibly arise at that scale?

Grand Unified Theories (GUTs) try to combine SU(3), SU(2), and U(1) into a single gauge group, where the SM arises from spontaneous symmetry breaking at the <u>GUT scale</u> of $\sim 10^{16}$ GeV.

ex. $SU(5) \longrightarrow SU(3) \times SU(2) \times U(1)$

The analogues of the W/Z are 12 new gauge bosons X, which can mix quarks and leptons.



, so $\frac{1}{\Lambda_6^2} = \frac{1}{m_X^2}$, and

$m_X > 10^{16}$ GeV.

$\Rightarrow$ observation of proton decay would tell us about enormously large energy scales!

"Back-of-the-envelope QFT"

While we're doing tilde-level estimates, it will be good to summarize some rules of thumb for estimating cross sections or decays.

- Phase space: for every extra particle in the final state, factor of $\frac{1}{4\pi^2}$ in rate. Comes from $d\Pi_n = \frac{d^3 p_n}{(2\pi)^3} \frac{1}{2E_n} d\Pi_{n-1}$. $\int d^3 p_n \sim 4\pi E_n^2$, matrix element has an extra power of E_n to have right dimensions. We saw this for $e^+e^- \to \mu^+\mu^-(\gamma)$: factor of e^2 in $|M|^2$ combined with $\frac{1}{4\pi^2}$ gives correction $\frac{e^2}{4\pi^2} = \mathcal{O}(\frac{\alpha}{\pi})$
- Loops: for every loop, add a factor $\frac{1}{4\pi^2}$ in $\mathcal{M}$. Here, the 4π's come from $\int \frac{d^4 k}{(2\pi)^4}$. We also saw this factor in $g-2$: $\frac{e^2}{4\pi^2} = \frac{\alpha}{\pi}$.

These quick order-of-magnitude estimates are good for guessing the answer <u>before</u> you start a long calculation! Also good for understanding the patterns in rare decay branching ratios [HW]