GR as an effective field theory

Outline:

I. GR from the bottom up – the unique Lagrangian for massless spin-2

II. GREFT; perihelion of Mercury from dimensional analysis alone!

I. Massless spin-2 particles have 2 polarizations. A Lorentz-invariant description requires such a particle to be embedded in the smallest Lorentz rep. containing spin-2: $j_1 = 1$ and $j_2 = 1 \Rightarrow (2j_1+1)(2j_2+1) = 9$, symmetric traceless tensors $h_{\mu\nu}$. (Actually, more convenient to start with trace included and project out, so $(0,0) \oplus (1,1)$.)

From these 10 components, we need to get down to 2 physical polarizations; this will strongly constrain the kinds of Lagrangians we can write down. It's easiest to see the algorithm by starting first with spin-1 as a warm-up.

A_μ: 4 components $\rightarrow$ 3 (massive) $\rightarrow$ 2 (massless)

Can split any 4-vector into transverse and longitudinal components: $A_\mu(x) = A_\mu^T(x) + \partial_\mu \pi(x)$ with $\partial^\mu A_\mu^T = 0$

(proof: $\partial^\mu A_\mu = \Box \pi$, so given A_μ, solve for π)

This decomposition is not unique but it exists; essentially defines Lorenz gauge for A^T.

Let's pretend we don't know about gauge invariance and want to find a sensible Lagrangian for A_μ.

Most general 2-derivative Lagrangian is (recall HW 3):

$$\mathcal{L} = a\, A^\mu \Box A_\mu + b\, A^\mu \partial_\mu \partial^\nu A_\nu + m^2 A^\mu A_\mu$$

(up to terms which vanish after integration by parts)

Plugging in $A = A^T + \partial\pi$. After some integration by parts,

$$\mathcal{L} = a\, A^{\mu T} \Box A_\mu^T + m^2 (A^{\mu T} A_\mu^T) - (a+b)\, \pi \Box^2 \pi - m^2 \pi \Box \pi$$

Claim: the theory is sick if $a+b \neq 0$.

Compute π propagator in momentum space: $\Box \to -k^2$, so

$$\Pi_\pi = \frac{1}{2} \frac{1}{-(a+b)k^4 + k^2 m^2} = \frac{1}{2m^2}\left[\frac{1}{k^2} - \frac{(a+b)}{(a+b)k^2 - m^2}\right]$$

$\Rightarrow$ π is actually two fields, but one has a wrong-sign propagator: a ghost. After quantization, leads to negative-norm states and non-unitary evolution: bad!

Only way out is to have $a+b=0$, which can be written in the more suggestive way with $a = -b = \frac{1}{2}$, $m^2 \to \frac{1}{2}m^2$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu$$

We can now restore a fictitious gauge symmetry (Stückelberg trick)

$$A_\mu^T \to A_\mu^T + \partial_\mu \alpha, \quad \pi \to \pi - \alpha$$

$$\Rightarrow \mathcal{L} = -\frac{1}{4} F_{\mu\nu}^T F^{\mu\nu T} + \frac{1}{2} m^2 (A_\mu^T + \partial_\mu \pi)^2.$$

Only two propagating modes in A_μ^T (transverse + gauge inv.), the third (longitudinal) mode is introduced explicitly with π:

Mass term for A $\Rightarrow$ kinetic term for π.

What about massless spin-1? If we try to set $m \to 0$, the kinetic term for π vanishes, but if A_μ couples to matter, $\mathcal{L} \supset A_\mu J^\mu$, then under gauge symmetry $\delta\mathcal{L} = \partial_\mu \pi J^\mu$

Bad things happen if interaction term is infinitely larger than kinetic term. Only way out: $\delta\mathcal{L} = 0$ up to total derivatives

$\Rightarrow \partial_\mu J^\mu = 0$. Massless spin-1 coupled to matter $\Rightarrow$ conserved J^μ.

OK, now to spin-2. Again, separate into transverse + long.:

$$h_{\mu\nu} = h^T_{\mu\nu} + \partial_\mu \pi_\nu + \partial_\nu \pi_\mu \quad \text{w/ } \partial^\mu h^T_{\mu\nu} = 0.$$ contains 4 d.o.f.

Also separate $\pi_\mu = \pi^T_\mu + \partial_\mu \pi^L$ w/ $\partial^\mu \pi^T_\mu = 0$. 1 d.o.f.

Most general 2-derivative quadratic Lagrangian is:

$$\mathcal{L} = a\, h_{\mu\nu} \Box h^{\mu\nu} + b\, h_{\mu\nu} \partial^\mu \partial^\alpha h^\nu{}_\alpha + c\, h \Box h + d\, h\, \partial^\mu \partial^\nu h_{\mu\nu}$$
$$+ m^2 (x\, h_{\mu\nu} h^{\mu\nu} + y\, h^2) \quad \text{w/ } h = h^\alpha{}_\alpha.$$

Same trick as before: after inserting transverse decomposition, look for terms involving π^L: $h_{\mu\nu} \supset 2\partial_\mu \partial_\nu \pi^L$, $h \supset 2\Box \pi^L$

$$m^2(x\, h_{\mu\nu} h^{\mu\nu} + y\, h^2) \supset m^2 (2x\, \partial_\mu \partial_\nu \pi^L \partial^\mu \partial^\nu \pi^L + 2y\, \Box \pi^L \Box \pi^L)$$
$$= 4m^2 (x+y)\, \pi^L \Box^2 \pi^L \quad \text{up to i.b.p.}$$

Same problem, same cure; need $x + y = 0$ to avoid ghosts.

Similar reasoning fixes relative coefficients of other terms:

$$\mathcal{L}_{FP} = \frac{1}{2} h_{\mu\nu} \Box h^{\mu\nu} - \frac{1}{2} h_{\mu\nu} \partial^\mu \partial^\alpha h^\nu{}_\alpha + h\, \partial^\mu \partial^\nu h_{\mu\nu} - \frac{1}{2} h \Box h + \frac{1}{2} m^2 (h_{\mu\nu} h^{\mu\nu} - h^2)$$

Fierz-Pauli Lagrangian for massive spin-2. Stückelberg trick: all terms but mass term are invt. under $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \alpha_\nu + \partial_\nu \alpha_\mu$, let $\pi_\mu \to \pi_\mu - \alpha_\mu$ and we are left with $10 - 4 - 4 = 2$ d.o.f's in $h^T_{\mu\nu}$ and $4 - 1 = 3$ d.o.f's in π_μ, leaving $2 + 3 = 5$ for massless spin-2.

If we blindly set $m=0$, we get

$$\mathcal{L} = \frac{1}{2} h_{\mu\nu} \Box h^{\mu\nu} - \frac{1}{2} h_{\mu\nu} \partial^\mu \partial^\alpha h^\nu{}_\alpha + h \partial^\mu \partial^\nu h_{\mu\nu} - \frac{1}{2} h \Box h$$

which is linearized vacuum Einstein-Hilbert; we are on the right track! But mass term gave π_μ^T a kinetic term. Need to make sure this disappears when $h_{\mu\nu}$ couples to other fields.

$$\mathcal{L} \supset h_{\mu\nu} T^{\mu\nu} \Rightarrow \delta\mathcal{L} = (\partial_\mu \pi_\nu + \partial_\nu \pi_\mu) T^{\mu\nu} \Rightarrow \partial_\mu T^{\mu\nu} = 0.$$

But this is not enough: consider $\mathcal{L}_1 = \frac{1}{2} h \phi$.

$\delta\mathcal{L}_1 = \partial^\mu \pi_\mu \phi$. If we let $\phi \to \phi + \pi_\mu \partial^\mu \phi$ and modify $\mathcal{L}$ to

$$\mathcal{L}_2 = \phi + \frac{1}{2} h\phi, \quad \delta\mathcal{L}_2 \supset \pi_\mu \partial^\mu \phi + \partial^\mu \pi_\mu \phi = \partial^\mu(\pi_\mu \phi) \to 0.$$

But now there are extra terms:

$$\delta\mathcal{L}_2 = \frac{1}{2} h \pi_\mu \partial^\mu \phi + (\partial^\mu \pi_\mu)(\pi_\nu \partial^\nu \phi)$$

To cancel these, need to modify transformation of h, which means adding more terms $\mathcal{L}_3 \supset h^2 \phi, \ldots$

Miraculously, this process converges!

$$\phi \to \phi(x+\pi), \quad h_{\mu\nu} \to (\eta_{\alpha\mu} + \partial_\alpha \pi_\mu)(\eta_{\beta\nu} + \partial_\beta \pi_\nu)\left[\eta^{\alpha\beta} + h^{\alpha\beta}(x+\pi)\right] - \eta_{\mu\nu}$$

In other words, a <u>general coordinate</u> transformation

$$\Rightarrow \mathcal{L} = M_{Pl}^2 \sqrt{-\det\left(\eta_{\mu\nu} + \frac{1}{M_{Pl}} h_{\mu\nu}\right)} \left(R\left[\eta_{\mu\nu} + \frac{1}{M_{Pl}} h_{\mu\nu}\right] + \mathcal{L}_m[\phi]\right)$$

(this is <u>not</u> trivial, but it's true)

$\Rightarrow$ GR is the <u>unique</u> theory of a massless spin-2 particle which couples to matter.

The factors of $\frac{1}{M_{Pl}}$ are for dimensional consistency: $[h_{\mu\nu}] = 1$, so this is just like the Chiral Lagrangian $F_\pi^2 \left[D_\mu \exp\left(\frac{\pi}{F_\pi}\right)\right]^2$

II. GR as an EFT.

Let's be schematic and ruthlessly suppress indices. Riemann tensor has two derivatives $R_{\mu\nu\alpha\beta} \sim \partial_\mu \partial_\nu \underbrace{\exp\left(\frac{1}{m_{pl}} h_{\alpha\beta}\right)}$,

analogous to $U = \exp\left(\frac{i}{F_\pi} \sigma^a \pi^a\right)$

$$\mathcal{L}_{EH} = M_{pl}^2 \, Tr[R_{\mu\nu}] \iff \mathcal{L}_{chiral} = F_\pi^2 \, Tr[DU^+ DU]$$

Each term has 2 derivatives and an infinite number of powers of h.

As with chiral Lagrangian, should write down all terms consistent with symmetry (in this case, diff invariance):

$$\mathcal{L} = \sqrt{\det(-g)} \, \Big(\underbrace{M_{pl}^2 R}_{\partial^2} + \underbrace{L_1 R^2 + L_2 R_{\mu\nu} R^{\mu\nu}}_{\partial^4} + L_3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + \ldots\Big)$$

total derivative, disappears in perturbation thy.

$$\mathcal{L} \sim \left(\tfrac{1}{2} h \Box h + \frac{1}{m_{pl}} \Box h^3 + \ldots\right) + L_i\left(\frac{1}{m_{pl}^2} h \Box^2 h + \frac{1}{m_{pl}^3} h \Box^2 h^2 + \ldots\right)$$

Just like chiral Lagrangian, this theory intrinsically contains higher-dimension terms even with only 2 derivatives: non-renormalizable

$\Rightarrow$ theory must break down (and needs a UV completion) at $E \sim m_{pl}$. But below that, perfectly predictive!

Example: let's look at effects of $\frac{1}{m_{pl}} \Box h^3$ term. We can use classical field perturbation theory:

equation of motion is $\Box h \sim \frac{1}{m_{pl}} \Box(h^2) - \frac{1}{m_{pl}} T$, where T is the energy-momentum tensor of a classical source.

Lowest-order solution is $h^{(0)} = \frac{-1}{m_{pl}} \frac{1}{\Box} T$; for $T = m \delta^3(r)$,

this is just the Newtonian potential $h^{(0)} = \frac{-m}{M_{pl}} \frac{1}{r}$.

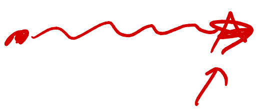
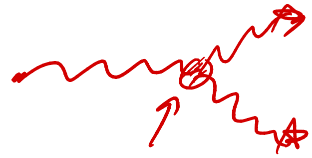
Perturbative solution:

$$\Box\left(h^{(0)}+h^{(1)}\right) = -\frac{1}{m_{pl}}T + \frac{1}{m_{pl}}\Box\left(h^{(0)}+h^{(1)}\right)^2 \quad \text{where } h^{(1)} = \mathcal{O}\left(\frac{1}{m_{pl}^3}\right)$$

$$\Box h^{(1)} = \frac{1}{m_{pl}}\Box\left(\frac{1}{m_{pl}^2}\frac{1}{\Box^2}T^2\right) + \mathcal{O}\left(\frac{1}{m_{pl}^4}\right)$$

$$\Rightarrow h^{(1)} = \frac{1}{m_{pl}^3}\frac{1}{\Box^2}T^2 \sim \frac{1}{m_{pl}}\left(\frac{m}{m_{pl}}\frac{1}{r}\right)^2$$

This is just the position-space classical version of Feynman diagrams:

$h^{(0)} =$  , $h^{(1)} =$ 

source   interaction

$$\frac{h^{(1)}}{h^{(0)}} \sim \frac{1}{m_{pl}}\frac{m}{m_{pl}}\frac{1}{r}.$$ Take $m = M_\odot$, $r =$ dist. btw. Sun and Mercury.

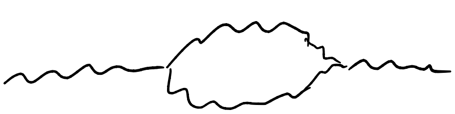
$$\frac{h^{(1)}}{h^{(0)}} \sim \frac{M_\odot}{m_{pl}}\frac{1}{m_{pl}r} \sim 10^{38}\frac{1}{10^{45}} \sim 10^{-7},$$ which is the perihelion shift!

$$\frac{43''/\text{century}}{2\pi/88\,\text{day}} = 0.8\times10^{-7}.$$

What about higher-order terms L_i? Can solve exactly w/ L_1 and L_2:

$$h(r) = \frac{m}{m_{pl}}\left[\frac{1}{r} - 128\pi^2\frac{L_1+L_2}{m_{pl}^2}\delta^3(r)+\ldots\right].$$ Short-range $\Rightarrow$ unobservable.

There are also genuine <u>quantum</u> effects:

 Corrects graviton propagator. Just like in non-abelian gauge theory, get a $\ln(-p^2)$ contribution which can't be cancelled by counterterms. Fourier-transform: $\ln(-p^2) \to \frac{1}{r^3}$ (c.f. Uehling potential in QED)

$$h(r) \sim \frac{m}{m_{pl}}\frac{1}{r}\Big[1 - \underbrace{\frac{m}{m_{pl}^2 r}}_{\text{"classical"}} - \underbrace{\frac{127}{30\pi^2}\frac{1}{m_{pl}^2r^2}}_{\text{quantum}} - \underbrace{128\pi^2\frac{L_1+L_2}{m_{pl}^2}\delta^3(r)}_{\text{UV completion predicts these}}+\ldots\Big]$$