Units in this class are “natural units” : \( \hbar = c = 1 \). In the SI system of units, there are three dimensionful quantities (mass, length, time), but relativity mixes length and time, and QM mixes energy and time from \( E=mc^2 \). So natural units make these conversions easy by having only one dimensionful quantity, mass (or energy, by \( E=mc^2 \)). Dimensions will be computed in powers of mass, and denoted \( [\ldots] \). 

Ex. \( [m] = 1 \) 
\( [E] = [mc^2] = [m] = 1 \) 
\( [T] = [\frac{E}{c}] = [E^{-1}] = -1 \) 
\( [L] = [cT] = [T] = -1 \)

An example in practice: 
\( F = q(E + \vec{v} \times \vec{B}) \), \( [q] = [v] = 0 \), and \( [E] = [ma] = [E^{-1}] = 2, \) so 
\( [E] = [B] = 2 \).

Two useful conversion factors to get back to SI: 
\( \hbar = 6.58 \times 10^{-32} \text{MeV} \cdot \text{s} \)
\( \hbar c = 197 \text{MeV} \cdot \text{fm} \)

Recall that Lorentz transformations are the set of linear coordinate transformations that leave the spacetime metric invariant. In this course, metric is \( g_{\mu\nu} = g^{\mu\nu} = \text{diag} (1,-1,-1,-1) \) so timelike 4-vectors have positive invariant mass.

A Lorentz “boost” along the z-axis by velocity \( |\vec{V}|<1 \) can be written as a matrix
\[
\Lambda = \begin{pmatrix}
Y & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
Y\beta & 0 & 0 & Y
\end{pmatrix}
\]
where \( Y = \frac{1}{\sqrt{1-V^2}} \).

In this class all transformations will be active, so acting on the 4-momentum of a particle at rest, \( \rho^\mu = (m, 0, 0, 0) \), gives \( \rho^\mu \to (\gamma m, 0, 0, \gamma \beta m) \). If \( \beta > 0 \), \( \rho^- \) is boosted to have \( \rho^- \).
We can extract a couple useful facts from this calculation:

• \( E = \gamma m \), so to find the Lorentz factor for a massive particle, just divide its energy by its mass.
• \( |\vec{p}| = \gamma \beta m \), so \( \beta = \frac{|\vec{p}|}{E} \). In this course we will almost never care about \( \beta \) and will use \( \gamma \) exclusively.

Recall \( p^2 = \hat{p} \cdot \hat{p} \equiv (p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2 \) is invariant; same in any frame. Comparing rest-frame \( p^\mu = (m, \vec{0}) \) to some other frame \( p'^\mu = (E, \vec{p}) \) gives \( E^2 = |\vec{p}|^2 + m^2 \) which we will use all the time.

Massless particles (e.g., photons) are described by lightlike 4-vectors with \( p^2 = 0 \), thus \( E = |\vec{p}| \) (and \( \beta = 1 \)).

An easy way to immediately see that a quantity is Lorentz-invariant is to use index notation. A Lorentz transformation \( \Lambda \) is a 4×4 matrix with entries \( \Lambda^u_v \), \( u, v = 0, 1, 2, 3 \)

\[
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\]

\( \Lambda \) labels rows, \( \nu \) labels columns.

Ex. \( \Lambda^0_0 = \Lambda^3_3 = \gamma \beta \), etc.

Greek indices run from 0 to 3, Latin indices \( i,j,k \), etc. run from 1 to 3.

Covariant vectors \( V^\mu \) transform by matrix multiplication:

\[
V^\mu \rightarrow \Lambda^\mu_\nu V_\nu \quad (\equiv \Lambda \cdot V, \text{contract top matrix index})
\]

Note Einstein summation convention: Sum over repeated upper/lower indices.

Contravariant vectors transform with the transpose of \( \Lambda \):

\[
W^\mu \rightarrow \Lambda^{\nu}_\mu W^{\nu} \quad (\equiv W \cdot \Lambda^T, \text{contract bottom matrix index})
\]

Can raise and lower indices (i.e., convert covariant to contravariant) by using the metric: \( V^\mu \equiv g^{\mu_\nu} V_\nu, \ W^\mu \equiv g_{\mu_\nu} W^{\nu} \). This is nice because we never have to keep track of transposes explicitly.
Lorentz transformations are defined to be those that preserve the metric: \[ \eta^{\mu\nu} = \Lambda^\mu_\alpha \Lambda^\nu_\beta \eta^{\alpha\beta} \] or \[ \eta = \Lambda^T \eta \Lambda. \]

This implies that any quantity with all indices contracted is a Lorentz scalar, i.e. invariant.

Example: \[ V^\mu W^\nu = \eta^{\mu\nu} V^\alpha W^\beta \Rightarrow \eta^{\mu\nu} = V^\mu W^\nu = V \eta W \]

Perform Lorentz transformation \( \Lambda \) on both \( V \) and \( W \):

\[ W^\mu V \rightarrow (W^{\nu^T}) \eta^{\nu\mu} = W (\Lambda^T \eta \Lambda) V = W \eta^{-1} V = W \eta V \]

Transposes and inverses are related by the metric preservation eqn:
\[ \Lambda^T \eta \Lambda = \eta \Rightarrow (\eta^{\nu^T} \nu \Lambda) = \eta \eta = 1, \text{ so } \Lambda^{-1} = \eta^{\nu^T} \eta \]

With indices, \( (\Lambda^{-1})^\mu_\nu = \eta^{\alpha\mu} \eta^{\beta\nu} \Lambda^\alpha_\beta \), but by the index raising/lowering rules, the RHS gets the same symbol \( \Lambda^\nu_\mu \), so we don't have to keep track of inverses either.

To be clear, this is just notational simplicity: if we wanted to evaluate components of the inverse transformation for our boost, we could do so explicitly: \( (\Lambda^{-1})^3_0 = \eta^{3\alpha} \eta^{0\alpha} \Lambda^\alpha_3 = \eta^{35} \eta^{05} \Lambda^3_0 = -V_3 \). But our notation means we don't have to distinguish between e.g. \( \Lambda^\nu_\mu \) and \( \Lambda^\mu_\nu \) as some texts do.

Check Lorentz invariance with index notation:
\[ V^\mu W^\nu \rightarrow \Lambda^\mu_\nu V^\alpha W^\beta = (\Lambda^{-1})^\mu_\nu \Lambda^\alpha_\beta V^\nu W^\rho = \tilde{\eta}^{\nu^T} \nu \Lambda^\alpha_\beta V^\nu W^\rho = V^\rho W^\rho \]

Tensors have more than one index: each lower index transforms with a factor of \( \Lambda \), each upper index w/ \( \Lambda^T \)

\[ e.g. \quad T_{\mu \nu} \rightarrow \Lambda^\alpha_\mu \Lambda^\beta_\nu T_{\alpha \beta} \]
\[ S_{\rho \sigma} \rightarrow \Lambda^\rho_\beta \Lambda^\sigma_\alpha S_{\beta \alpha} \]

With index notation, we know that a quantity like \( T_{\mu \nu} T^{\mu \nu} \) is invariant under Lorentz transformations just by looking at it.
One last piece of notation:

\[ \frac{\partial}{\partial x^m} \equiv (\partial_0, \partial_1, \partial_2, \partial_3) \] is "naturally" a covariant vector, while \( x^m \) is "naturally" contravariant.

\[ J^\wedge_\alpha \equiv g^{\alpha \nu} \partial_\nu \partial_\alpha = (\partial_0)^2 - (\partial_1)^2 - (\partial_2)^2 - (\partial_3)^2 \] is called the d'Alembertian and is often denoted \( \Box \).