Observations (many!) tell us physics is invariant with respect to Lorentz transformations. Therefore, our goal is to describe elementary particles in a Lorentz-invariant way.

An elementary particle is an irreducible representation of the Poincaré group — a semidirect product of the Lorentz group and the group of spacetime translations — classified by its two Casimir invariants, mass and spin. If the particle is charged, it is an irreducible representation of an additional internal symmetry, global or gauged.

Over the next 3 weeks, we will learn what all these words mean.

Group: a collection $G$ of objects $\Lambda$ with an associative multiplication rule satisfying

a) identity: $I \Lambda = \Lambda I = \Lambda$ for any $\Lambda \in G$ and some specific $I \in G$

b) inverse: for any $\Lambda \in G$, there exists $\Lambda^{-1}$ in $G$ such that
$$\Lambda \Lambda^{-1} = \Lambda^{-1} \Lambda = I$$

c) closure: if $\Lambda_1, \Lambda_2 \in G$, then $\Lambda_1 \Lambda_2 \in G$.

Note: multiplication is not necessarily commutative: $\Lambda_1 \Lambda_2 \neq \Lambda_2 \Lambda_1$ in general.

Representation: a map $G \to \text{Mat}^{n \times n}$. Elements of $G$ can then act on vectors in the vector space $\mathbb{R}^n$ by matrix multiplication.
Claim: Lorentz transformations form a group, which we call $SO(3,1)$.

Two ways to see this:

1) explicit calculation (compose two boosts and see you can get another boost, etc.)

2) be more abstract and clever

Define $SO(3,1)$ as the set of $4 \times 4$ real matrices $\Lambda$ satisfying $[\Lambda^T \gamma \Lambda = \gamma]$, with $\gamma = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$.

Let's take an example to verify that this makes sense. [in-class exercise]

$$\Lambda_x = \begin{pmatrix} \gamma & \gamma b & 0 & 0 \\ \gamma b & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \Lambda_x^T \Lambda \gamma \Lambda = \gamma_x.$$  Multiplication by $\gamma$ on the left multiplies rows by diagonal elements of $\gamma$, so

$$\Lambda^T (\gamma \Lambda) = \begin{pmatrix} \gamma & \gamma b & 0 & 0 \\ \gamma b & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \gamma^2 (1-\beta^2) & 0 & 0 & 0 \\ 0 & -\gamma^2 (1-\beta^2) & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and since $\gamma^2 = \frac{1}{1-\beta^2}$, the RHS is $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \gamma$.

Verify group properties from the definition:

- Identity: take $I = \Lambda_{id}$. Then $I^T \gamma I = \gamma$, so $I \in SO(3,1)$.
- Inverse: the matrix inverse $\Lambda^{-1}$ is an inverse to $\Lambda$ as long as $\Lambda^{-1} \in SO(3,1)$, so we need to show $(\Lambda^{-1})^T \gamma \Lambda^{-1} = \gamma$. Start with inverting defining relationship: $(\Lambda^T \gamma \Lambda)^{-1} = \gamma^{-1}$

$$\Rightarrow \Lambda^{-1} \gamma (\Lambda^{-1})^T = \gamma \text{ since } \gamma^{-1} = \gamma.$$
Want $(A^{-1})^T$ on left, so left-multiply both sides by $y A$ and right-multiply by $y A^{-1}$:

$$(y A)(A^{-1}) (A A^{-1}) = (y A)(y A^{-1}) \Rightarrow (A^{-1})^T y A^{-1} = y$$

since $(A^T)^{-1} = (A^{-1})^T$.

• Closure: 

These 4×4 matrices are also a representation of the group, since they were used to define the group, we call it the defining representation. It acts on 4-vectors $x^v$ as $A^v u^v$.

What about other representations?

• Trivial representation: All elements of $SO(3,1)$ map to the number 1. This is the "do-nothing" representation and acts on scalars (numbers).

• What about acting on 2-component vectors? 3-component?

To do this systematically, we need the concept of Lie algebras. These are another mathematical collection of objects obtained from a group by looking at group elements infinitesimally close to the identity.

Let's try writing $A = I + \epsilon X$ and expand to first order in $\epsilon$.

$\gamma = (I + \epsilon X)^T \gamma (I + \epsilon X) = I \gamma I + \epsilon (X^T \gamma + \gamma X) + \Theta(\epsilon^2)$

$\Rightarrow X^T \gamma = -\gamma X$ defines Lie algebra $\mathfrak{so}(3,1)$

Up to multiplication by $y^v$, this looks like the condition for an antisymmetric 4×4 matrix, which has $\frac{4 \times 3}{2} = 6$ independent parameters. Thus the dimension of $\mathfrak{so}(3,1)$ (and $SO(3,1)$) is 6.
Unlike $SO(3,1)$, $\mathfrak{so}(3,1)$ does not have a multiplication rule.

It is, however, a vector space: if $X, Y \in \mathfrak{so}(3,1)$, then $aX + bY \in \mathfrak{so}(3,1)$ for any real numbers $a, b$.

It has one additional ingredient, called the Lie bracket:

If $X, Y \in \mathfrak{so}(3,1)$, then $[X, Y] = XY - YX \in \mathfrak{so}(3,1)$

Proof: $([X, Y])^\gamma \equiv (XY - YX)^\gamma$

$$= Y^\gamma X^\gamma - X^\gamma Y^\gamma$$

$$= Y^\gamma (-\gamma X) - X^\gamma (-\gamma Y)$$

$$= \gamma (XY - YX)$$

$$= -\gamma [X, Y]$$

Since taking brackets keeps us in the Lie algebra, we can choose a basis $T_i$ and write $[T_i, T_j] = f_{ijk} T_k$, where $f_{ijk}$ are called structure constants, and the whole equation is a commutation relation.

For $\mathfrak{so}(3,1)$, it's easiest to split the basis into infinitesimal boosts and infinitesimal rotations, and to allow ourselves complex coefficients.

Let $J = (J_x, J_y, J_z)$ be infinitesimal rotations around $x, y, z$ axes respectively.  Ex. $J_x = i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \gamma \\ \gamma \\ \gamma \end{pmatrix}$

$K \equiv (K_x, K_y, K_z)$ are infinitesimal boosts along $x, y, z$.

Ex. $K_x = i \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Commutation relations: $[J_i, J_j] = i\epsilon_{ijk} J_k$, $[K_i, K_j] = -i \epsilon_{ijk} J_k$, $[J_i, K_j] = i\epsilon_{ijk} K_k$

Look familiar?  Two boosts give a rotation.  [CHW]
The fact that J and K get mixed with each other is annoying. But we have one more trick up our sleeve: define a new basis
\[ J^+ = \frac{J + iK}{2}, \quad J^- = \frac{J - iK}{2} \]

In this basis, the commutation relations are
\[ [J^+_i, J^-_j] = i \epsilon_{ijk} J^+_k, \quad [J^+_i, J^-_j] = i \epsilon_{ijk} J^-_k, \quad [J^+_i, J^-_j] = 0 \]

Two identical copies of the same Lie algebra which don't mix!

So representation theory of \( SO(3,1) \) boils down to representation theory of \( J^+ \) and \( J^- \)

But you already know the answer from quantum mechanics!

2d rep: \( J^i = \sigma_i \), Pauli matrices (spin \(-\frac{1}{2}\))

3d rep: \( A_i = \text{infinitesimal 3d rotations} \) (spin \(-1\))

Using raising and lowering operators, can have any half-integer spin representation of dimension \( 2j + 1 \)

\[ \Rightarrow \] Pick a half-integer \( j \) labeling \( J^- \) and another half-integer \( j' \) for \( J^+ \), and this defines a rep. of the Lorentz group \((J, j')\) of dimension \((2j+1)(2j'+1)\). Some examples:

\( j \quad j' \quad 0 \quad \frac{1}{2} \quad 1 \)

- 0: Scalars
- \( \frac{1}{2} \): Left-handed fermions
- 1: Right-handed fermions
- 2: 4-vectors
- 3: Gravitons

This is all we will need to describe the Standard Model.
The world has more symmetries than just Lorentz transformations: translations in space and time. These translations form a group too; \( \mathbb{R}^+ \), since we can write \( x^m \rightarrow x^m + \lambda^m \) as a 4-vector.

Combine translations with rotations and boosts? Have to be a bit careful because translations and rotations don't commute. Correct structure is a semi-direct product: if \( \alpha \) and \( \beta \) are translations, and \( \Lambda_1, \Lambda_2 \) are Lorentz transformations,

\[
(\alpha, \Lambda_1) \cdot (\beta, \Lambda_2) = (\alpha + \Lambda_1 \beta, \Lambda_1 \Lambda_2)
\]

Usual multiplication law from last lecture apply Lorentz transf. \( \Lambda_1 \) to 4-vector \( x \), then translate by \( \alpha \)

\[
\Rightarrow \text{this is a group, } \mathbb{R}^+ \times SO(3,1)
\]

Let's revisit the Lie algebra of the Lorentz group, but now with indices.

\[
\gamma = 1 + \epsilon X \quad \Rightarrow \quad \gamma^\mu = \delta^\mu + \epsilon \gamma^\mu \quad (w \text{ are entries of matrix } X)
\]

\[
\gamma^\mu \gamma = \gamma \quad \Rightarrow \quad \gamma^\mu \gamma^\nu \gamma_{\nu\rho} = \gamma_{\mu\rho}
\]

Plug in expansion of \( \gamma \), isolate \( \Theta(6) \) terms as before:

\[
(\delta^\mu + \epsilon \gamma^\mu)(\delta^\nu + \epsilon \gamma^\nu)\gamma_{\rho\sigma} = \gamma_{\mu\nu}
\]

\[
\gamma_{\mu\nu} + \epsilon (\delta^\mu \gamma^\nu + \delta^\nu \gamma^\mu)\gamma_{\rho\sigma} + \Theta(6^2) = \gamma_{\mu\nu}
\]

(Use \( \gamma_{\rho\sigma} \) to lower indices) \( \epsilon (\delta^\mu \gamma^\nu + \delta^\nu \gamma^\mu) = 0 \)

\[
\Rightarrow \quad \gamma_{\mu\nu} + \gamma_{\nu\mu} = 0
\]

\[ \text{, so } \gamma_{\mu\nu} \text{ is an antisymmetric tensor w/ 6 independent components: 3 boosts and 3 rotations.} \]