NOTE: $\omega_{\mu\nu}$ is only antisymmetric with both its indices lowered! From now on in the course index heights come with important minus signs.

A general infinitesimal Lorentz transformation can be written

$$X = \frac{-i}{2} \omega_{\mu\nu} M^{\mu\nu} = -i (\omega_0, M^{00} + \omega_0, M^{03} + \omega_3, M^{03} + \omega_3, M^{33}),$$

where

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & \beta_1 & \beta_2 & \beta_3 \\ -\beta_1 & 0 & \theta_3 & -\theta_2 \\ -\beta_2 & -\theta_3 & 0 & \theta_1 \\ -\beta_3 & \theta_2 & -\theta_1 & 0 \end{pmatrix}, \quad M^{\mu\nu} = \begin{pmatrix} 0 & -K_1 & -K_2 & -K_3 \\ K_1 & 0 & J_3 & -J_2 \\ K_2 & -J_3 & 0 & J_1 \\ K_3 & J_2 & -J_1 & 0 \end{pmatrix},$$

($\omega_0 = \beta_0, \ \omega_{ij} = E_{ijk} \theta_k$)

\(\beta = (\beta_0, \beta_1, \beta_2, \beta_3)\) is an infinitesimal boost vector, and \(\theta = (\theta_0, \theta_1, \theta_2)\) is an infinitesimal rotation about the axis \(\theta\). This lets us write an infinitesimal transformation in a way where there are no minus sign ambiguities. \(\omega_{\mu\nu} = \beta_0, \ \omega_{\mu\nu} = E_{ijk} \theta_k\)

\[X = -i \theta \cdot J + i \beta \cdot K\]

Note minus sign compared to Schwartz!

Our convention is the usual RH rule for rotations.

A convenient way to write the entries of \(M^{\mu\nu}\), each of which is a 4x4 matrix, is \((M^{\mu\nu})^x_y = i (\gamma^x \delta^y_h - \gamma^y \delta^x_h)\) where \(x, y\) label the matrix indices.

\[\begin{align*}
\gamma^0 &= i (\gamma^0 \delta^0_h - \gamma^h \delta^0_0) \\
&\quad + 1 \text{ if } x=0, y=1 \quad \quad + 1 \text{ if } x=1, y=0
\end{align*}\]

Using this form we can compute the commutator for all generators:

$$[M^{\mu\nu}, M^{\rho\sigma}]^x_y = (M^{\mu\nu})^x_y (M^{\rho\sigma})^y_h - (M^{\rho\sigma})^y_h (M^{\mu\nu})^x_y$$

\[
\begin{align*}
&= - (\gamma^x \delta^0_h - \gamma^0 \delta^x_h)(\gamma^0 \delta^y_h - \gamma^y \delta^0_0) + (\gamma^y \delta^0_h - \gamma^0 \delta^y_h)(\gamma^x \delta^h_0 - \gamma^h \delta^x_0) \\
&= -\gamma^x \gamma^0 \delta^0_h + \gamma^y \gamma^0 \delta^x_h + (3 \text{ similar}) \\
&= -i \gamma^x \gamma^0 (M^{\rho\sigma})^y_h + (3 \text{ similar}) \\
&= -i \gamma^x \gamma^0 (M^{\rho\sigma})^y_h + (3 \text{ similar}) \\
&\Rightarrow |M^{\mu\nu}, M^{\rho\sigma}| = i (\gamma^x \gamma^0 M^{\rho\sigma} + \gamma^x \gamma^y M^{\rho\sigma} - \gamma^y \gamma^0 M^{\rho\sigma} - \gamma^y \gamma^y M^{\rho\sigma})
\end{align*}
\]
Now let's include transformations to get the whole Poincare group. 

\[ x^\mu \rightarrow x^\mu + \Lambda^\mu \] can be implemented as a matrix with one extra entry:

\[
\begin{pmatrix}
1 & x^0 & x^1 & x^2 & x^3 \\
0 & 1 & x^1 & x^2 & x^3 \\
0 & 0 & 1 & x^2 & x^3 \\
0 & 0 & 0 & 1 & x^3
\end{pmatrix}
\begin{pmatrix}
x^0 \\
x^1 \\
x^2 \\
x^3
\end{pmatrix}
= 
\begin{pmatrix}
x^0 + \Lambda^0 \\
x^1 + \Lambda^1 \\
x^2 + \Lambda^2 \\
x^3 + \Lambda^3
\end{pmatrix}
\] (this is called an affine transformation)

So a general Poincare element (Lorentz + translation) can be represented as:

\[(\lambda, \Lambda) = \begin{pmatrix} 1 & i & \lambda_1 & \lambda_2 & \lambda_3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \]

\[(\lambda_1, \Lambda_1), (\lambda_2, \Lambda_2) = \begin{pmatrix} \lambda_1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Lambda_1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 + \lambda_2 & \Lambda_1 + \Lambda_2 \\ 0 & 1 \end{pmatrix} \]

Infinite translation is still a vector, let's call it \( P^\mu \).

\[ P^\mu = -i \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad P^\mu = +i \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \]

Note signs! In other words, \((P^\mu)^\alpha = -i \sigma^\alpha \) where \( \alpha \) is the row index of the 5x5 matrix.

\[[ P^\mu, P^\nu ] = 0 \quad [HW]\]

One last commutation relation to compute:

\[[ M^{\mu \nu}, P^\alpha ]^\alpha = \begin{pmatrix} (m^{\mu \nu})^\alpha & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & i (P^\mu)^\alpha \\ 0 & i (P^\nu)^\alpha \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & (m^{\mu \nu})^\alpha (P^\mu)^\alpha \\ 0 & (m^{\mu \nu})^\alpha (P^\nu)^\alpha \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \]

\[ P^\alpha \text{ transforms like a 4-vector, as it should} \]

So the commutator is a pure translation (Lorentz part is 0).
Compute the coefficient:

\[ i(\gamma^\mu \gamma^\alpha \gamma^\beta - \gamma^\mu \gamma^\alpha \gamma^\beta)(-i\eta^{\alpha\beta}) = i(\gamma^\nu (-i\eta^{\alpha\beta}) - \gamma^{\nu\beta}(-i\gamma^\nu)) \]

\[ = i(\gamma^\nu \gamma^\nu - \gamma^{\nu\nu} \gamma^\nu) \]

\[ \Rightarrow [M^{\mu\nu}, \gamma^{\alpha\beta}] = i(\gamma^\nu \gamma^\nu - \gamma^{\nu\nu} \gamma^\nu) \]

We now have the complete commutation relations for the Lie algebra of the Poincaré group:

\[
\begin{align*}
[M^{\mu\nu}, M^{\rho\sigma}] &= i(\gamma^{\nu\rho} M^{\mu\sigma} + \gamma^{\nu\sigma} M^{\mu\rho} - \gamma^{\rho\sigma} M^{\mu\nu} - \gamma^{\mu\nu} M^{\rho\sigma}) \\
[M^{\mu\nu}, P^{\rho}] &= i(\gamma^{\nu\rho} P^{\mu} - \gamma^{\rho\mu} P^{\nu}) \\
[P^{\mu}, P^{\nu}] &= 0
\end{align*}
\]

Note that while we derived these using a particular $5 \times 5$ representation of the Lie algebra, they hold in general as abstract operator relations. Just like with the Lorentz group, we will now systematically construct the representations of this group.

**Casimir operators**

Now that we have the algebra, what can we do with it? If we find an object that commutes with all generators, a theorem from math tells us it must be proportional to the identity operator on any irreducible representation; this is called a Casimir operator.

Irreducible $\leftrightarrow$ can't write as block-diagonal like

\[
\begin{pmatrix}
R_1 & 0 \\
0 & R_n
\end{pmatrix}
\]
Here's one Casimir operator: \( p^2 = p^\mu p_\mu \). Proof:

\[
[p^\mu, p^\nu] = 0 \quad \text{since all } p\text{'s commute.}
\]

\[
[p^\mu, M^{\mu\nu}] = p^\nu [p_\nu, M^{\mu\nu}] + [p^\mu, M^{\mu\nu}] p_\nu \quad \text{(using } [A;B,C] = A[B,C] + [A,C]B) \]

\[
= p^\nu (-i(\gamma^\nu p^\mu - \gamma^\mu p^\nu)) - i(\gamma^\nu p^\mu - \gamma^\mu p^\nu) p_\nu \]

\[
= i(p^\nu p^\mu - p^\mu p^\nu) + i(p^\mu p^\nu - p^\nu p^\mu) = 0
\]

(which had to be true: \( M^{\mu\nu} \) is antisymmetric in \( \mu, \nu \), and since \( [M, p] \propto p \), could only have a commutator like \( pp \) which is symmetric in \( \mu, \nu \))

Thus on an irreducible rep., \( p^2 \) acts as a constant times the identity operator. Let's call the constant \( m^2 \); we will soon identify it with the physical (squared) mass of a particle.

The Poincaré algebra has a second Casimir, but it's a bit less transparent. Let's define

\[
W_\sigma = \frac{1}{2} \epsilon_{\mu\nu\rho} M^{\mu\nu} p^\rho
\]

(Pauli-Lubanski: pseudovector)

\( \epsilon_{\mu\nu\rho} \) is the totally antisymmetric tensor with \( \epsilon_{0123} = -1 \).

We will see that \( W \) is related to a particle's spin. First, some useful observations:

- \( W \) is orthogonal to \( p \): \( W_\sigma p^\sigma \propto \epsilon_{\mu\nu\rho} p^\mu p^\nu p^\rho = 0 \) by antisymmetry of \( \epsilon \).

- \( W \) and \( p \) commute, so we can label reps. by both their eigenvals.

\[
[W_\sigma, p^\nu] = \frac{1}{2} \epsilon_{\mu\nu\rho} [M^{\mu\nu}, p^\rho] = \frac{1}{2} \epsilon_{\mu\nu\rho} \left( M^{\mu\nu}[\rho^\rho, p^\rho] + [M^{\mu\nu}, p^\rho] p^\rho \right)
\]

\[
= \frac{1}{2} \epsilon_{\mu\nu\rho} (\gamma^{\nu} p^{\mu} - \gamma^{\mu} p^{\nu}) p^\rho
\]

\( = 0 \), again by antisymmetry.
Now, consider some state $|k^+\rangle$ which is an eigenvector of $p^m$ w/eigenvalue $k^m$. We will see next week that such states describe particles of definite momentum. $p^+$ acts as $k^+k^m = m^\pm$, so indeed, for a massive particle, $p^+$ acts as the identity on all states $|k^+\rangle$ related by Lorentz transformations.

But to a frame where $k^+ = (m, 0, 0, 0)$, so $p^+ |k^+\rangle = m |k^+\rangle$, $p^|k^+\rangle = 0$. Then $W(|k^+\rangle) = \frac{1}{2} E_{ijk} M^{ik} p^j |k^+\rangle = m (\frac{1}{2} E_{ijk} M^{ik}) |k^+\rangle = -m J^+ |k^+\rangle$

As you recall from QM, $J^\pm \equiv J^x \pm J^y = s(s+1)$ is indeed a multiple of the identity with coefficient given by the particle’s spin $s$, so the same should hold true for $W^\pm = -(\bar{w} \cdot \bar{w}) = -m^2 J^\pm J^\mp$.

Note: this only works if $m \geq 0$!! Will come back to $m = 0$.

Claim: $W^\pm W^\mp$ is a Casimir, i.e. commutes with all $p^m$ and $M^{mn}$

Proof: we have already shown $[W, p] = 0$, so clearly $[W^\pm, p] = 0$.

But $W^\mp$ is Lorentz-invariant (no free indices), so the action of an infinitesimal Lorentz transformation must vanish: $[W^\mp, M^{mn}] = 0$.

If this argument is too slick for you, for HW you will check explicitly that $[W^\pm, M^{mn}] = 0$ using the Poincaré algebra.

Physical interpretation of Casimirs:

Recall from the second lecture that $J^+ = \frac{J^x + i J^y}{2}$, $J^- = \frac{J^x - i J^y}{2}$

$\Rightarrow J = J^+ + J^-$

Reps of Lorentz group are labeled by half-integer spins $j_1, j_2$, so this is like adding spins in QM! $J$ can have spins $j = j_1 - j_2, j_1 - j_2 + 1, \ldots, j_1 + j_2$, with $J^2 = j(j+1)$
But $W^2$ is a Casimir operator so it only takes one value on each irreducible representation, which one?

Some easy cases: $(0, 0)$ rep has $j_1 = j_2 = 0$ so $j = 0$: these are spin-0 particles (scalars)

$(\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$ reps have $j_1 = \frac{1}{2}$ and $j_2 = 0$ or vice-versa: again, only one possible value of $j$, $j = \frac{1}{2}$, so these are spin-$\frac{1}{2}$ particles

More interesting:

$(\frac{1}{2}, \frac{1}{2})$ rep has $j_1 = j_2 = \frac{1}{2}$, so $j = 1$ or 0. In QFT, this will describe spin-1 particles, but we will need an additional constraint in the equations of motion to project out the $j = 0$ component.

What about massless particles? $p^2 = 0$, so we can't go to a frame where $K^0 = (m, 0, 0, 0)$. The best we can do is to take $k^0 = k$ and pick a direction since $|k^\mu| = k^0$; take $k^\mu = (k, 0, 0, k)$.

Can show that $W$ generates the set of transformations which leave $k^\mu$ fixed (this is known as the little group). This is clear for $m \neq 0$ since $J$ generates rotations which leave the zeroth component alone and don't affect $k^\mu = 0$.

For $m = 0$, things are more subtle. Clearly rotations in the $xy$-plane preserve $k^z = k_3$, so $W_1 |k\rangle = W_3 |k\rangle = 0$. But there is actually a combination of a boost and a rotation that also preserves $k^\mu$. Note that $W_\mu p^\mu = 0 \Rightarrow k(W_0 + W_3) |k\rangle = 0$, so $W_0 |k\rangle = -W_3 |k\rangle$. Can also show $[W_0, W_3] |k\rangle = 0$. [HW]

Therefore, $W^2 |k\rangle = (W_0)^2 - (W_3)^2 |k\rangle = 0$, so eigenvalues of $W^2$ alone aren't enough to tell us about spin.
If we raise an index, $\omega^0\ell = W^3/k$, so $\omega^{-1}\ell = \lambda \rho^{-1}\ell$ for some $\lambda$.

Consider $W_0 = \frac{1}{2} \epsilon_{i,j,k} M^{ij} \rho^k = -\frac{1}{2} \epsilon_{0,i,j,k} M^{ij} \rho^k = + \vec{\rho} \cdot \vec{p} = \lambda \rho_0$.

Since $\rho_0\ell = |\vec{p}|\ell$ for massless particles, solve for $\lambda$:

$\lambda = \frac{\vec{J} \cdot \vec{p}}{|\vec{p}|} = \vec{J} \cdot \vec{p}$. This is a new spin quantum number called helicity: projection of spin along direction of motion.

It is lorentz-invariant for massless particles! $\vec{J} \cdot \vec{p} = J_3$ is quantized in half-integers, therefore so is $\lambda$. Examples:

- $(0,0)$ rep: $J_3 = 0$ so $\lambda = 0 \Rightarrow \text{spin}-0$.
- $(\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$ reps: $J_3 = \frac{1}{2} \sigma$ so $J_3 = \pm \frac{1}{2}$, and $\lambda = \pm \frac{1}{2}$. $\lambda > 0$ means “spin-up along direction of motion,” which we call right-handed. For $m \geq 0$, this property is invariant under boosts.

- $(\frac{1}{2}, \frac{1}{2})$ rep: $\lambda = -1$, $0$ (x2), or $+1 \Rightarrow \text{spin}-1$, but $\lambda = 0$ states are unphysical.

Compared to $m > 0$, there is an extra $\lambda = 0$ state which we will have to get rid of with gauge invariance.