Note the advantages of index notation here:

If a Lagrangian has all indices contracted, it's invariant under Lorentz transformations.

e.g. $\partial_\mu \Phi \, \partial_\nu \Phi$ is not Lorentz-invariant, but $\partial_\mu \Phi \, \partial^\mu \Phi$ is.

- **U(1) symmetry:** $\Phi \to e^{i\alpha\alpha}\Phi$. We also require $\Phi^\dagger \to e^{-i\alpha\alpha}\Phi^\dagger$

So that $\Phi^\dagger = (\Phi^*)^T$ before and after transformation

$\Rightarrow$ any terms that have an equal number of Φ and $\Phi^\dagger$ are invariant, as long as α is a constant.

$$\partial_\mu \Phi^\dagger \, \partial_\nu \Phi \longrightarrow (e^{-i\alpha\alpha}\partial_\mu \Phi^\dagger)(e^{i\alpha\alpha}\partial_\nu \Phi) = \partial_\mu \Phi^\dagger \partial_\nu \Phi$$

$$(\Phi^\dagger \Phi)^2 = (e^{-i\alpha\alpha}\Phi^\dagger e^{i\alpha\alpha}\Phi)^2 = (\Phi^\dagger\Phi)^2, \text{ etc.}$$

Just like with Lorentz/Poincaré, we can consider infinitesimal transformations:

$$e^{i\alpha\alpha} = 1 + i\alpha\alpha + \cdots, \text{ so } \Phi \to (1+i\alpha\alpha)\Phi \text{ or } \delta\Phi = i\alpha\alpha\Phi$$

This is a convenient calculational trick, so let's apply it:

$$\delta(\Phi^\dagger\Phi) = (\delta\Phi^\dagger)\Phi + \Phi^\dagger(\delta\Phi) = (-i\alpha\alpha\Phi^\dagger)\Phi + \Phi^\dagger(+i\alpha\alpha\Phi) = 0$$

the "variation operator" δ distributes over products

If $\delta(\cdots) = 0$, that term is invariant under the symmetry.

- **SU(2) symmetry:** $\Phi \to e^{i\alpha^a\sigma^a/2}\Phi$. Recall the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For real parameters α^a $(a=1,2,3)$, $\dfrac{i\alpha^a\sigma^a}{2} = \dfrac{i}{2}\begin{pmatrix} \alpha^3 & \alpha^1-i\alpha^2 \\ \alpha^1+i\alpha^2 & -\alpha^3 \end{pmatrix} \equiv iX \quad \in \mathfrak{su}(2)$

$$M \equiv e^{iX} = 1 + iX + \frac{(iX)^2}{2!} + \cdots \quad \in SU(2)$$

If X is Hermitian, M is <u>unitary</u> (#HW)

Why SU(2) instead of U(2)?

Suppose we diagonalize M so $\det M = \prod_i \lambda_i$ (product of eigenvalues)

$$\log(\det M) = \log\left(\prod_i \lambda_i\right) = \sum_i \log \lambda_i = \text{Tr}(\log M)$$

But Tr and $\det$ are both basis-independent so they hold for any M, in particular $M = e^{iX}$

If $\text{Tr}(X) = 0$, then $\text{Tr}(\log M) = \text{Tr}(iX) = 0$, so $\log(\det M) = 0$, $\det M = 1$

$\Rightarrow$ traceless, Hermitian X exponentiate to unitary matrices M with determinant 1.

Here, Pauli matrices are 2×2, so they exponentiate to the group SU(2) (indeed, they are the Lie algebra of SU(2), i.e. the set of infinitesimal transformations)

Back to Lagrangian: again, any terms with an equal number of $\underline{\Phi}$ and $\underline{\Phi}^+$ are invariant.

Proof: $\delta \underline{\Phi} = i \frac{\alpha^a \sigma^a}{2} \underline{\Phi}$, $\delta \underline{\Phi}^+ = \left(\frac{i\alpha^a \sigma^a}{2} \underline{\Phi}\right)^+ = \underline{\Phi}^+ \left(\frac{-i\alpha^a \sigma^a}{2}\right)$

(σ^a are Hermitian)

$$\delta(\underline{\Phi}^+\underline{\Phi}) = (\delta\underline{\Phi}^+)\underline{\Phi} + \underline{\Phi}^+(\delta\underline{\Phi}) = \underline{\Phi}^+\left(\frac{-i\alpha^a \sigma^a}{2}\right)\underline{\Phi} + \underline{\Phi}^+\left(\frac{i\alpha^a \sigma^a}{2}\right)\underline{\Phi}$$

$$= \underline{\Phi}^+\left(\frac{-i\alpha^a \sigma^a + i\alpha^a \sigma^a}{2}\right)\underline{\Phi}$$

$$= 0$$

What does $\delta\underline{\Phi}$ do to the fields in $\underline{\Phi}$? Write out some examples:

$\alpha = (1,0,0)$ $\delta\underline{\Phi} = \frac{i\sigma^1}{2}\underline{\Phi} = \begin{pmatrix} 0 & \frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}\begin{pmatrix} \phi_1 + i\phi_2 \\ \ell_1 + i\ell_2 \end{pmatrix} = \begin{pmatrix} -\frac{\ell_2}{2} + \frac{i\ell_1}{2} \\ -\frac{\phi_2}{2} + \frac{i\phi_1}{2} \end{pmatrix}$

i.e. $\delta\phi_1 = -\frac{\ell_2}{2}$, $\delta\phi_2 = \frac{\ell_1}{2}$, $\delta\ell_1 = -\frac{\phi_2}{2}$, $\delta\ell_2 = \frac{\phi_1}{2}$

<u>mixes</u> fields among one another (i.e. "rearranges the labels" on field operators)

We have now identified all the spacetime and global (i.e. constant) symmetries of $\mathcal{L}$. To wrap up, a little dimensional analysis.

Action S should be dimensionless in natural units, since it appears in the path integral as a phase e^{iS}.

$$\left[\int d^4x \, \mathcal{L} \right] = 0 \implies [d^4x] + [\mathcal{L}] = 0$$

$$-4 + [\mathcal{L}] = 0$$

$$\boxed{[\mathcal{L}] = 4} \quad \longleftarrow \text{\color{red}{the key to understanding 90\% of QFT in 4 spacetime dimensions!}}$$

We saw that for a scalar field, a mass term can be written as $\mathcal{L} \underset{\text{\color{red}{"contains"}}}{\supset} m^2 \Phi^\dagger \Phi$. So with $[m] = 1$, we must have $\boxed{[\Phi] = 1}$

$[\partial_\mu] = \left[\frac{\partial}{\partial x^\mu} \right] = \left[\frac{1}{x^\mu} \right] = 1$, so $[\partial_\mu \Phi] = 2$ and the derivative ("kinetic") term also has dimension 4: $[g^{\mu\nu} \partial_\mu \Phi^\dagger \partial_\nu \Phi] = 4$.

$[(\Phi^\dagger \Phi)^2] = 4$, but what about $(\Phi^\dagger \Phi)^3$? To put this in a Lagrangian must include a dimensionful constant $\left[\frac{1}{\Lambda^2} \right] = -2$ such that

$\frac{1}{\Lambda^2}(\Phi^\dagger \Phi)^3$ has dimension 4. This means that something interesting happens at energies Λ; more on this in the last 2 weeks of the course!

Gauge invariance and spin-1

Recall our scalar Lagrangian from last time:

$$\mathcal{L}[\Phi] = \partial_\mu \Phi^\dagger \partial^\mu \Phi - m^2 \Phi^\dagger \Phi - \lambda (\Phi^\dagger \Phi)^2$$

We saw that $\delta \Phi = iQ\alpha \Phi$ was a symmetry. What if we let $\alpha = \alpha(x^\mu)$ depend on spacetime position? This is a local transformation because it's a different action at each point, in contrast to global which is the same everywhere.

The spacetime dependence doesn't affect the second and third terms, which remain invariant, but it does change the first one:

$$\delta(\partial_\mu \Phi^\dagger \partial^\mu \Phi) = \partial_\mu \delta\Phi^\dagger \partial^\mu \Phi + \partial_\mu \Phi^\dagger \partial^\mu(\delta\Phi)$$

$$= \partial_\mu(-iQ\alpha(x)\Phi^\dagger)\partial^\mu \Phi + \partial_\mu \Phi^\dagger \partial^\mu(iQ\alpha(x)\Phi)$$

$$= -iQ\partial_\mu\alpha \, \Phi^\dagger \partial^\mu \Phi + iQ\partial^\mu\alpha \, \partial_\mu\Phi^\dagger \Phi$$

   Not invariant anymore!

We can fix this with a trick: swap out all instances of ∂_μ with

$$D_\mu \equiv \partial_\mu - igQA_\mu(x) \quad \text{(covariant derivative)} \text{ where } g \text{ is called a coupling constant}$$

We define A_μ to have the transformation rule $\boxed{A_\mu \to A_\mu + \frac{1}{g}\partial_\mu\alpha}$ for both finite and infinitesimal α

Then $D_\mu \Phi = \partial_\mu \Phi - igQA_\mu \Phi$ transforms as

$$D_\mu \Phi \to \partial_\mu(e^{iQ\alpha}\Phi) - igQ(A_\mu + \frac{1}{g}\partial_\mu\alpha)e^{iQ\alpha}\Phi$$

$$= iQ\partial_\mu\alpha \, e^{iQ\alpha}\Phi + e^{iQ\alpha}\partial_\mu\Phi - igQA_\mu e^{iQ\alpha}\Phi - iQ\partial_\mu\alpha \, e^{iQ\alpha}\Phi$$

$$= e^{iQ\alpha}(\partial_\mu\Phi - igQA_\mu\Phi) = e^{iQ\alpha}D_\mu\Phi$$

Transformation of A_μ cancels extra term from derivative of local symmetry parameter

$$\Rightarrow \quad D_\mu\Phi^\dagger D^\mu\Phi \to (e^{-iQ\alpha}D_\mu\Phi^\dagger)(e^{iQ\alpha}D^\mu\Phi) = D_\mu\Phi^\dagger D^\mu\Phi,$$
    invariant under local symmetry

So, we can promote a global symmetry $\Phi \Rightarrow e^{iQ\alpha}\Phi$ to a local symmetry $\Phi \rightarrow e^{iQ\alpha(x)}\Phi$, at the cost of introducing another field A_μ which has its own non-homogeneous transformation rule $A_\mu \rightarrow A_\mu + \frac{1}{g}\partial_\mu \alpha$.

Why in the world would we do this?

- Turns out this is the correct way to incorporate interactions with spin-1 fields: A_μ will be the photon, and Q is the the electric charge. (The coupling constant is $g = \sqrt{4\pi\alpha}$ where $\alpha \approx 1/137$ is the fine-structure constant you saw in QM.)

- In fact, this transformation rule for A_μ is required for a consistent, unitary theory of a massless spin-1 particle: invariance under this local transformation is known as gauge invariance.

Let's put Φ aside for now and just consider what form the Lagrangian for A_μ must take.

- Lorentz invariance: A_μ is a Lorentz vector, so $A_\mu(x) \rightarrow \Lambda^\nu_\mu A_\nu(\Lambda^{-1}x)$. So the "principle of contracted indices" holds: $A_\mu A^\mu$ is Lorentz-invariant, as is $(\partial_\mu A_\nu)(\partial^\mu A^\nu)$, etc.

- Gauge invariance: we want $\mathcal{L}$ to be invariant under $A_\mu \Rightarrow A_\mu + \frac{1}{g}\partial_\mu \alpha$.

Try writing down a mass term:
$$\delta\left(\frac{1}{2}m^2 A_\mu A^\mu\right) = \frac{1}{2}m^2 \left(\delta A_\mu A^\mu + A_\mu \delta A^\mu\right)$$
$$= \frac{m^2}{g}\partial_\mu \alpha A^\mu \neq 0$$

Surprise! A mass term is not allowed by gauge invariance.

What about terms with derivatives? Something like $\partial_\mu A_\nu$ will pick up $\partial_\mu \partial_\nu \alpha$. Can cancel this with a compensating term $\partial_\nu \partial_\mu \alpha$, which comes from $\partial_\nu A_\mu$. This leads to $\mathcal{L}_A = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$

conventional

$F_{\mu\nu}$, field strength tensor

With $A_\mu = (\phi, \vec{A})$, the electromagnetic potentials, you will find that $\mathcal{L}$ is none other than the Maxwell Lagrangian, $\frac{1}{2}(\vec{E}^2 - \vec{B}^2)$.

But the photon has 2 polarizations, i.e. 2 independent components of A_μ, which is a 4-vector. How do we get rid of the 2 extraneous components? Two-step process:

1. Note that A^0 has no time derivatives: $\partial_0 A_0$ never appears in Lagrangian, so its equation of motion doesn't involve time. Therefore A_0 is not a propagating degree of freedom: this follows immediately from writing $\mathcal{L}[F_{\mu\nu}]$. Can solve for A^0 in terms of $\vec{A}$ $\Rightarrow$ 3 components left.

2. Choose a gauge, for example $\vec{\nabla} \cdot \vec{A} = 0$. Solve for one component of $\vec{A}$ in terms of the other two, and what's left are the two propagating degrees of freedom, whose equations of motion are $\Box A^{(1,2)} = 0$.

The counting is fairly straightforward as above, but not Lorentz invariance; under a Lorentz transformation, A^0 mixes with $\vec{A}$, $\vec{\nabla} \cdot \vec{A} = 0$ is not preserved, etc.

_ _ _ _ _

Repeat the above analysis using unitary representations of the Lorentz group.

A 4-vector A_μ must have some Hilbert space representation $|A_\mu\rangle$, so we can write a state $|\psi\rangle$ as a linear combination of the components:

$$|\psi\rangle = c_0 |A_0\rangle + c_1 |A_1\rangle + c_2 |A_2\rangle + c_3 |A_3\rangle$$

This state must have positive norm:

$$\langle \psi | \psi \rangle = |c_0|^2 + |c_1|^2 + |c_2|^2 + |c_3|^2 > 0.$$

But if the components of A_μ change under a Lorentz transformation, we can change the norm, which is bad; the Lorentz transformation matrices are not unitary!

Alternatively, we could redefine the norm to be Lorentz-invariant,

$$\langle \psi | \psi \rangle = |c_0|^2 - |c_1|^2 - |c_2|^2 - |c_3|^2, \text{ but this is not positive definite!}$$

Solution in two steps: (1) use *fields* as the representation, which do have unitary (infinite-dimensional) representations, and (2) project out the wrong-sign component. Since vectors live in the $(\frac{1}{2}, \frac{1}{2})$ representation, which has $j=0$ and $j=1$ components, this is equivalent to projecting out the $j=0$ component, leaving $j=1$ as appropriate for spin-1.

Write A_μ in Fourier space:
$$A_\mu(x) = \int \frac{d^4 p}{(2\pi)^4} \epsilon_\mu(p) e^{-ip\cdot x}$$

↙ momentum-dependent polarization vector

A Lorentz transformation will act on this field as
$$A_\mu(x) \longrightarrow \Lambda^\nu_\mu A_\nu(\Lambda^{-1} x) = \int \frac{d^4 p}{(2\pi)^4} \underbrace{\Lambda^\nu_\mu \epsilon_\nu(p)} e^{-ip\cdot(\Lambda^{-1}x)}$$

polarization vectors rotate, but p_μ (a dummy integration variable) does <u>not</u>.

This explains why we pick eigenstates of P^μ before defining action of W_μ.

Use equations of motion to count independent polarizations:
$$\Box A_\mu - \partial_\mu(\partial^\nu A_\nu) = 0 \quad \text{\color{green}(HW)}$$

Choose a gauge such that $\partial^\nu A_\nu = 0$. (Can always do this: if $\partial^\nu A_\nu = X$, take $A_\nu \to A_\nu + \frac{1}{g}\partial_\nu \lambda$, $\partial^\nu A_\nu \to X + \frac{1}{g}\partial^2 \lambda$. Solve for λ to cancel X.)

$\Rightarrow$ in Fourier space, $p^2 = 0$ and $p\cdot \epsilon = 0$. The latter is an <u>algebraic</u> constraint which is Lorentz-invariant, so it projects out spin-0 as desired. Reduces <u>four</u> polarizations $\epsilon_\mu^{(0)} = (1,0,0,0)$, $\epsilon_\mu^{(1)} = (0,1,0,0)$, ---

to <u>three</u>. But we have one more gauge transformation left!

Can still have $A_\mu = \partial_\mu \lambda$ consistent with $\partial^\mu A_\mu = 0$ if $\partial^2 \lambda = 0$. In this case, A_μ is gauge-equivalent to 0 (or pure <u>gauge</u>) and not physical. After Fourier-transforming, this means the polarization proportional to 4-momentum ($\epsilon_\mu \propto p_\mu$) is <u>unphysical</u>.