We are thus left with <u>two</u> independent polarization vectors: in a frame where $p_\mu = (E, 0, 0, E)$, they are

$$\epsilon_\mu^{(1)} = (0, 1, 0, 0)$$
$$\epsilon_\mu^{(2)} = (0, 0, 1, 0)$$

} linear polarization

or

$$\epsilon_\mu^{(L)} = \frac{1}{\sqrt{2}}(0, 1, -i, 0)$$
$$\epsilon_\mu^{(R)} = \frac{1}{\sqrt{2}}(0, 1, i, 0)$$

} circular polarization

In QFT, these polarization vectors represent physical states, so we can take linear combinations of them:

e.g. $|\epsilon\rangle = c_1|1\rangle + c_2|2\rangle$. <u>Define</u> $\langle i|j\rangle = -\epsilon_\mu^{(i)}\epsilon^{*\mu(j)}$

$$\langle\epsilon|\epsilon\rangle = |c_1|^2\langle 1|1\rangle + |c_2|^2\langle 2|2\rangle + c_1^* c_2 \langle 1|2\rangle + c_1 c_2^* \langle 2|1\rangle$$

$\langle 1|1\rangle = -(\epsilon_\mu^1)^* \epsilon^{1\mu} = 1$

$\langle 1|2\rangle = 0$ since ϵ_μ^1 and ϵ_μ^2 are orthogonal

$$= |c_1|^2 + |c_2|^2$$

This inner product is Lorentz-invariant because the basis vectors change under Lorentz, but <u>not</u> $|c|^2$! Moreover, gauge invariance let us get rid of the states with non-positive norm:

$\epsilon_\mu^{(0)} = (1, 0, 0, 0) \Rightarrow \langle 0|0\rangle = -1$, bad!

$\epsilon_\mu^{(f)} = (1, 0, 0, 1) \Rightarrow \langle f|f\rangle = 0$, unphysical (cancels out of any computation)

(forward, or longitudinal, polarization)

Including the Lagrangian for A_μ, our spin-0 and spin-1 Lagrangian is now

$$\mathcal{L} = |D_\mu \Phi|^2 - m^2 \Phi^\dagger \Phi - \lambda(\Phi^\dagger\Phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Note: $[A_\mu] = [\partial_\mu] = 1$ from covariant derivative, so $[F_{\mu\nu}F^{\mu\nu}] = 4$, as required.

The derivative term in the Lagrangian for Φ with only the global symmetry, $\partial_\mu \Phi^\dagger \partial^\mu \Phi$, gave rise to the equations of motion for non-interacting (free) scalar fields.

Once promoted to a covariant derivative, $|D_\mu \Phi|^2$ contains interactions between Φ and A_μ.

$$|D_\mu \Phi|^2 = (\partial_\mu \Phi^\dagger + igQA_\mu \Phi^\dagger)(\partial^\mu \Phi - igQA^\mu \Phi)$$

$$= \partial_\mu \Phi^\dagger \partial^\mu \Phi - A_\mu \Big(-igQ\underbrace{\big(\Phi^\dagger \partial^\mu \Phi - \partial^\mu \Phi^\dagger \Phi\big)}\Big) + g^2 Q^2 A_\mu A^\mu |\Phi|^2$$

in QM, this would be the probability current for the wavefunction. In QFT, it's literally the electric current for a charged scalar particle.

$\Rightarrow \mathcal{L}$ contains $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu J^\mu$, which is exactly how you would write Maxwell's equations with an external source $J^\mu = (\rho, \vec{J})$! So Φ sources currents, which create $\vec{E}$ and $\vec{B}$ fields from A_μ, which back-reacts on Φ. These coupled equations are impossible to solve exactly, so starting in 2 weeks we will use perturbation theory in the coupling strength gQ to approximate the solutions.

Nonabelian gauge fields (very briefly!)

What if we tried the same trick with the SU(2) symmetry?

We want the Lagrangian to be invariant under the local symmetry $\Phi \to e^{i\alpha^a(x)\tau^a}\Phi$ where $\tau^a \equiv \frac{\sigma^a}{2}$ $(a=1,2,3)$. Guess a covariant derivative: $\boxed{D_\mu \Phi = \partial_\mu \Phi - i g A_\mu^a \tau^a \Phi}$. This time, we now need three spin-1 fields A_μ^a, one for each τ.

Will postpone proof for later, but the correct transformation rules are $\boxed{\delta A_\mu = \frac{1}{g}\partial_\mu \alpha + i[\alpha, A_\mu]}$ (matrix commutator)

or in components, $\delta A_\mu^a = \frac{1}{g}\partial_\mu \alpha^a - \epsilon^{abc}\alpha^b A_\mu^c$ (recall commutation relations for Pauli matrices, $[\sigma^a,\sigma^b] = 2i\epsilon^{abc}\sigma^c$)

The corresponding non-abelian field strength (a 2x2 matrix-valued Lorentz tensor) is $F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu) - ig[A_\mu, A_\nu]$ ← extra term because Pauli matrices don't commute!

A clever way to write this:

$D_\mu = \partial_\mu - igA_\mu$ (abstract covariant derivative operator)

$$[D_\mu, D_\nu] = (\partial_\mu - igA_\mu)(\partial_\nu - igA_\nu) - (\partial_\nu - igA_\nu)(\partial_\mu - igA_\mu)$$
$$= \cancel{\partial_\mu\partial_\nu} - ig\partial_\mu A_\nu - ig\cancel{A_\nu\partial_\mu} - ig\cancel{A_\mu\partial_\nu} - g^2 A_\mu A_\nu$$
$$- \cancel{\partial_\nu\partial_\mu} + ig\partial_\nu A_\mu + ig\cancel{A_\mu\partial_\nu} + ig\cancel{A_\nu\partial_\mu} + g^2 A_\nu A_\mu$$
$$= -ig\left(\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]\right)$$
$$= -igF_{\mu\nu}$$

Can show (★HW) that $\delta F_{\mu\nu} = [i\alpha, F_{\mu\nu}]$, so $F_{\mu\nu}$ itself is not gauge invariant. However,

$$\underbrace{\delta(F_{\mu\nu}\cdot F^{\mu\nu})}_{\text{matrix product and Einstein summation}} = \delta F_{\mu\nu}\cdot F^{\mu\nu} + F_{\mu\nu}\cdot\delta F^{\mu\nu} = [i\alpha, F_{\mu\nu}]F^{\mu\nu} + F_{\mu\nu}[i\alpha, F^{\mu\nu}]$$
$$= i\alpha F_{\mu\nu}F^{\mu\nu} - \cancel{F_{\mu\nu}(i\alpha)F^{\mu\nu}} + \cancel{F_{\mu\nu}(i\alpha)F^{\mu\nu}} - F_{\mu\nu}F^{\mu\nu}i\alpha$$

One last trick: $\mathrm{Tr}(ABC\cdots) = \mathrm{Tr}(BC\cdots A)$. Trace is cyclically invariant, so by taking the trace, we can cancel the remaining terms and get a gauge-invariant object.

$$\mathcal{L}_{SU(2)} = -\frac{1}{2}\mathrm{Tr}(F_{\mu\nu}\cdot F^{\mu\nu})$$

SU(2) indices

$$= -\frac{1}{4}\left(F_{\mu\nu}^{\underline{1}}F^{\mu\nu\,\underline{1}} + F_{\mu\nu}^{\underline{2}}F^{\mu\nu\,\underline{2}} + F_{\mu\nu}^{\underline{3}}F^{\mu\nu\,\underline{3}}\right) \text{ because}$$

$$\mathrm{Tr}((\tau^1)^2) = \mathrm{Tr}((\tau^2)^2) = \mathrm{Tr}((\tau_3)^2) = \frac{1}{4}\mathrm{Tr}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \frac{1}{2}.$$

This looks just like 3 copies of the Lagrangian for the U(1) gauge field, but hidden inside $F_{\mu\nu}F^{\mu\nu}$ are interaction terms, i.e.

$$F_{\mu\nu}^1 F^{\mu\nu\,1} \supset A_\mu^2 A_\nu^3 \partial^\mu A^{1\nu}$$

The gauge field interacts with itself!

Let's switch to standard notation and call the SU(2) gauge field W and the U(1) gauge field B. We can also relabel the coupling $gQ \to g'Y$ (will see why next week):

$$D_\mu\Phi = (\partial_\mu - ig'YB_\mu - igW_\mu^a\tau^a)\Phi$$

$$\boxed{\mathcal{L}_{\Phi,\,\text{gauged}} = |D_\mu\Phi|^2 - m^2\Phi^\dagger\Phi - \lambda(\Phi^\dagger\Phi)^2 - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}W_{\mu\nu}^a W^{\mu\nu\,a}}$$

This completes one part of our desired classification: a Lagrangian describing a spin-0 particle of mass m invariant under Poincaré transformations and the (gauged) internal symmetries U(1) and SU(2). This description requires us to pick the representations of U(1) and SU(2) on Φ: the former is parametrized by a number Y, and the latter is a choice of representation matrices, where we have chosen the 2-dimensional rep. using the Pauli matrices. The Lagrangian has Φ and W self-interactions, as well as Φ-W and Φ-B interactions.

Massive spin-1 fields

As we saw, a mass term for a vector field is not gauge invariant. However, there are several massive spin-1 particles in nature, which are either composite particles (the ρ meson, for example) or which acquire a mass through the Higgs mechanism (the W and Z gauge bosons). So, we should understand what their Lagrangians should look like without assuming any gauge invariance conditions.

Luckily, the story is still quite simple. We still need to get rid of 1 extraneous degree of freedom, and this will restrict the form of the Lagrangian.

We want a Lagrangian whose equations of motion will yield $(\Box + m^2)A_\mu = 0$ in order to satisfy the relativistic dispersion $p^2 = m^2$. So we can have quadratic terms with 0 or 2 derivatives. The most general such Lagrangian is

$\mathcal{L} = \frac{a}{2} A^\mu \Box A_\mu + \frac{b}{2} A^\mu \partial_\mu \partial^\nu A_\nu + \frac{1}{2} m^2 A^\mu A_\mu$ with a, b, m arbitrary coefficients. (Note that $[\mathcal{L}] = 4$ if $[A] = 1$, a and b are dimensionless, and $[m] = 1$.)

The equations of motion are [HW]

$$a \Box A_\mu + b \partial_\mu \partial^\nu A_\nu + m^2 A_\mu = 0.$$

Take ∂^μ of this to get

$$((a+b)\Box + m^2)(\partial^\mu A_\mu) = 0.$$

We are on the right track if we can enforce $\partial^\mu A_\mu = 0$: this is a scalar (i.e. spin-0) constraint so it projects out $j=0$ as desired. To do this, take $a=1$, $b=-1$:

$$\mathcal{L} = \frac{1}{2}A^\mu \Box A_\mu - \frac{1}{2}A^\mu \partial_\mu \partial_\nu A^\nu + \frac{1}{2}m^2 A^\mu A_\mu$$

$$= -\frac{1}{2}\left(\partial^\nu A^\mu \partial_\nu A_\mu - \partial^\nu A^\mu \partial_\mu A_\nu\right) + \frac{1}{2}m^2 A^\mu A_\mu \quad \text{(integrating by parts)}$$

$$= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) + \frac{1}{2}m^2 A^\mu A_\mu \quad \text{(rearranging)}$$

$$= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A^\mu A_\mu \quad \leftarrow \text{Proca (massive spin-1) Lagrangian}$$

The field strength $F_{\mu\nu}$ just appeared without having to invoke gauge invariance! The equations of motion are now

$$(\Box + m^2)A_\mu = 0 \quad \text{and} \quad \partial^\mu A_\mu = 0.$$

We can now find the 3 linearly-independent polarization vectors as before, but now in a frame where $p^\mu = (m, 0, 0, 0)$ since the Poincaré Casimir $p^2 = m^2$.

In Fourier space, have $p^2 = m^2$ and $p \cdot \epsilon = 0$. So can take $\epsilon^1_\mu = (0,1,0,0)$, $\epsilon^2_\mu = (0,0,1,0)$, and $\epsilon^L_\mu = (0,0,0,1)$. These satisfy $\epsilon^* \cdot \epsilon = -1$ as did the massless polarizations, and they are all physical.

In a boosted frame with $p^\mu = (E, 0, 0, p_z)$ $(p_z^2 = E^2 - m^2)$, we have

$$\epsilon^1_\mu = (0,1,0,0), \quad \epsilon^2_\mu = (0,0,1,0), \quad \epsilon^L_\mu = \left(\frac{p_z}{m}, 0, 0, \frac{E}{m}\right).$$

The third polarization is called longitudinal because it has a spatial component along the direction of motion.

Note that for ultra-relativistic energies $E \gg m$,

$$\epsilon^L_\mu \rightarrow \frac{E}{m}(1,0,0,1),$$

This will cause problems in QFT, and is why massive spin-1 must either be composite or arise from a Higgs mechanism.